Combinatorial Bethe ansatz and Generalized periodic box-ball system

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Abstract: We reformulate the Kerov-Kirillov-Reshetikhin (KKR) map in the combinatorial Bethe ansatz from paths to rigged configurations by introducing local energy distribution in crystal base theory. Combined with an earlier result on the inverse map, it completes the crystal interpretation of the KKR bijection for $U_q(\widehat{\mathfrak{sl}}_2)$. As an application, we solve an integrable cellular automaton, a higher spin generalization of the periodic box-ball system, by an inverse scattering method and obtain the solution of the initial value problem in terms of the ultradiscrete Riemann theta function.

1. Introduction

The Kerov-Kirillov-Reshetikhin (KKR) bijection [12, 13] is a combinatorial version of the Bethe ansatz. It gives a one to one correspondence between rigged configurations and highest paths, which are combinatorial analogues of the Bethe roots and the associated Bethe vectors in integrable spin chains[1]. The relevant problem of state counting stemmed from Bethe's original work [3], was developed further in the KKR theory, and has been formulated as the $X = M$ conjecture for arbitrary affine Lie algebra [16]. See [18, 30] for a recent status.

In [20, 28], the KKR map ϕ^{-1} from rigged configurations to highest paths was identified with a certain composition of combinatorial R in crystal base theory [10, 11, 14]. It provided a long sought representation theoretical meaning with ϕ^{-1} and opened a connection with the integrable cellular automata called the box-ball system [33, 34] and its generalizations [17, 7, 15]. They are identifed with solvable vertex models [2] associated with the quantum group U_q at $q = 0$. In this context, the KKR theory is regarded as the inverse scattering formalism of the generalized box-ball systems, where the rigged configurations and ϕ^{-1} play the roles of scattering data and inverse scattering transform, respectively. Precise descriptions are available either in Proposition 2.6 in [20], section 3.2 and appendix E in [25], and Lemma B.9 in this paper.

In this paper we study two closely related problems concerning $U_q(\widehat{\mathfrak{sl}}_2)$ case. In the first part (section 2), we give a crystal theoretical interpretation of the opposite KKR map ϕ from paths to rigged configurations. It is done by introducing the *local energy distribution* of paths, which provides a bird's-eye view of the whole combinatorial procedures involved in the KKR algorithm. In terms of generalized box-ball systems, ϕ is a direct scattering map and separates the dynamical degrees of freedom into action-angle variables, which are amplitudes and phase of solitons. The local energy distribution makes it possible to grasp these data from a global viewpoint. See Example 2.4. Together with the earlier result on ϕ^{-1}, we complete the crystal interpretation of the KKR bijection $\phi^{\pm 1}$ for $U_q(\widehat{\mathfrak{sl}}_2)$.

The results mentioned so far are concerned with generalized box-ball systems on (semi) infinite lattice. In the second part of the paper (section 3), we launch the inverse scattering formalism in the periodic case. This was achieved in [21] for the simplest spin $\frac{1}{2}$ system called the periodic box-ball system [40, 39], and subsequently in [26]. Here we treat the

[1] The original KKR bijection concerns semistandard tableaux rather than highest paths. The KKR bijection in this paper is to be understood as the composition of the original one with the Robinson-Schensted-Knuth correspondence between semistandard tableaux and highest paths.

general spin $\frac{s}{2}$ case based on the crystal base theory. Here is an example of time evolution (T_4) of an $s = 3$ case.

$$
\begin{array}{rl}
t=0: & 122\cdot 122\cdot 112\cdot 112\cdot 111\cdot 122\cdot 111\cdot 111\cdot 112\\
t=1: & 112\cdot 112\cdot 122\cdot 122\cdot 112\cdot 111\cdot 122\cdot 111\cdot 111\\
t=2: & 111\cdot 112\cdot 112\cdot 112\cdot 122\cdot 122\cdot 111\cdot 122\cdot 111\\
t=3: & 111\cdot 111\cdot 112\cdot 112\cdot 112\cdot 112\cdot 222\cdot 111\cdot 122\\
t=4: & 122\cdot 111\cdot 111\cdot 112\cdot 112\cdot 112\cdot 111\cdot 222\cdot 112\\
t=5: & 112\cdot 222\cdot 111\cdot 111\cdot 112\cdot 112\cdot 112\cdot 111\cdot 122\\
t=6: & 122\cdot 111\cdot 222\cdot 112\cdot 111\cdot 112\cdot 112\cdot 112\cdot 111\\
t=7: & 111\cdot 122\cdot 111\cdot 122\cdot 122\cdot 111\cdot 112\cdot 112\cdot 112\\
t=8: & 112\cdot 111\cdot 122\cdot 111\cdot 112\cdot 222\cdot 111\cdot 112\cdot 112\\
t=9: & 112\cdot 112\cdot 111\cdot 122\cdot 111\cdot 111\cdot 222\cdot 112\cdot 112\\
t=10: & 112\cdot 112\cdot 112\cdot 111\cdot 122\cdot 111\cdot 111\cdot 122\cdot 122\\
t=11: & 122\cdot 122\cdot 112\cdot 112\cdot 111\cdot 122\cdot 111\cdot 111\cdot 112
\end{array}
$$

A local spin $\frac{s}{2}$ state is an s array of 1 and 2 which are arranged not to decrease to the right. Each local state is regarded as a capacity s box. Local states, say $111, 112, 122$ and 222 for $s = 3$, represent an empty box and those containing $1, 2$ and 3 balls, respectively. An array of such local states are called paths. The above paths are of length 9.

A path of length L can naturally be viewed as an element of $B_s^{\otimes L}$, the tensor product of the crystal B_s of the s-fold symmetric tensor representation of $U_q(\widehat{\mathfrak{sl}}_2)$. A wealth of notions and combinatorial operations on B_s are provided by the crystal base theory. We make use of them to characterize a certain class of paths that are invariant under extended affine Weyl group $\widehat{W}(A_1^{(1)})$ and the commuting family of invertible time evolutions $\{T_l\}$. This is an important non-trivial step characteristic to the $s > 1$ situation. We introduce action-angle variables which correspond to those paths bijectively and linearize the dynamics. These features are integrated in Theorem 3.11. As corollaries of it, generic period and a counting formula of the paths are obtained in terms of conserved quantities in (3.28) and (3.26), respectively. For example (3.28) tells that the period of the above paths under T_4 is indeed 11. (Notice that the $t = 0$ and $t = 11$ paths are the same.) These results agree with the conjecture in the most general setting [22]. The initial value problem is solved either by a combinatorial algorithm or by an explicit formula (3.36), (3.34) involving the ultradiscrete Riemann theta function (3.30), generalizing the $s = 1$ results in [23, 24]. These expressions follow rather straightforwardly from the ultradiscrete tau function studied in [25]. For the background idea of ultra-discretization and relevant issues in tropical geometry, see [35] and [27].

Several characteristic features in quasi-periodic solutions to soliton equations [4, 6] will be demonstrated in the ultradiscrete setting. In particular our action-angle variables live in the set (3.15) which is an ultradiscrete analogue of the Jacobi variety. For a reduced case (3.29) with $s = 1$, the underlying tropical hyperelliptic curve has been identified recently [9]. The action-angle variables are essentially solutions of the string center equation, which is a version of the Bethe equation at $q = 0$ [19]. In this sense, the inverse scattering formalism in this paper connects the Bethe ansätze at $q = 1$ [13, 12] and $q = 0$ [19] to the algebraic geometry techniques of soliton theory at a combinatorial level.

Our crystal interpretation of the KKR map ϕ has stemmed from an attempt to formulate the direct scattering map in the generalized periodic box-ball system. In fact, we will show in section 3.3 that the idea of local energy distribution is efficient also in the periodic setting.

The paper is organized as follows. In section 2, the KKR map ϕ is identified with a procedure based on the local energy distribution in Theorem 2.2. We illustrate it along a few instructive examples. The proof will be given in [29]. Section 3 is devoted to the generalized periodic box-ball system. Section 4 is a summary. Appendix A recalls the basic

facts on crystal base theory [10, 11, 14]. Appendix B is an exposition of the KKR bijection including the non-highest case [5, 30].

2. Local energy distribution and the KKR bijection

In this section, we reformulate the combinatorial procedure of the KKR map ϕ in terms of the energy functions of crystal base theory. See Appendix A for the basic facts on crystal base theory. Consider the relation

$$a \otimes b_1 \simeq b'_1 \otimes a'$$

and the energy function $e_1 = H(a \otimes b_1)$ under the combinatorial R. We depict them by the vertex diagram:

$$\begin{array}{ccc} & b_1 & \\ & \vert & \\ a & \overset{e_1}{\text{———}}\!\!+\!\!\text{———} & a'. \\ & \vert & \\ & b'_1 & \end{array}$$

Successive applications of the combinatorial R

$$a \otimes b_1 \otimes b_2 \simeq b'_1 \otimes a' \otimes b_2 \simeq b'_1 \otimes b'_2 \otimes a'',$$

with $e_2 = H(a' \otimes b_2)$ is expressed by joining two vertices:

$$\begin{array}{ccccc} & b_1 & & b_2 & \\ & \vert & & \vert & \\ a & \overset{e_1}{\text{———}}\!\!+\!\!\text{———} & a' & \overset{e_2}{\text{———}}\!\!+\!\!\text{———} & a''. \\ & \vert & & \vert & \\ & b'_1 & & b'_2 & \end{array}$$

Given a path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L$, its local energy $\mathcal{E}_{l,j}$ is defined by $\mathcal{E}_{l,j} := H(u_l^{(j-1)} \otimes b_j)$, where $u_l^{(j-1)}$ is specified by the following diagram with the convention $u_l^{(0)} = u_l$ (A.6).

$$\begin{array}{ccccccc} & b_1 & & b_2 & & & b_L & \\ & \vert & & \vert & & & \vert & \\ u_l & \overset{\mathcal{E}_{l,1}}{\text{———}}\!\!+\!\!\text{———} & u_l^{(1)} & \overset{\mathcal{E}_{l,2}}{\text{———}}\!\!+\!\!\text{———} & u_l^{(2)} \cdots\cdots\cdots & u_l^{(L-1)} & \overset{\mathcal{E}_{l,L}}{\text{———}}\!\!+\!\!\text{———} & u_l^{(L)} \\ & \vert & & \vert & & & \vert & \\ & b'_1 & & b'_2 & & & b'_L & \end{array}$$

We set $\mathcal{E}_{0,j} = 0$ for all $1 \leq j \leq L$. We define $\mathcal{T}_l$ and $\mathcal{E}_l$ by $\mathcal{T}_l(b) = b'_1 \otimes b'_2 \otimes \cdots \otimes b'_L$ and

$$\mathcal{E}_l := \sum_{j=1}^{L} \mathcal{E}_{l,j}. \tag{2.1}$$

In other words, $u_l[0] \otimes b \overset{R}{\simeq} \mathcal{T}_l(b) \otimes u_l^{(L)}[\mathcal{E}_l]$, where we have omitted modes for b and $\mathcal{T}_l(b)$.

Given a path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L$ $(b_i \in B_{\lambda_i})$, we always have $u_l^{(L+\Lambda)} = u_l$ for any l for a modified path $b' = b \otimes \boxed{1}^{\otimes \Lambda}$ if $\Lambda > \lambda_1 + \cdots + \lambda_L$. In such a circumstance, $\mathcal{E}_l(\mathcal{T}_k(b')) = \mathcal{E}_l(b')$ is known to hold (Theorem 3.2 of [7], section 3.4 of [15]). Namely the sum $\mathcal{E}_l$ is a conserved quantity of the box-ball system on semi-infinite lattice. Here we need more detailed information such as $\mathcal{E}_{l,j}$.

Lemma 2.1. *For a path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L$, we have $\mathcal{E}_{l,j} - \mathcal{E}_{l-1,j} = 0$ or 1, for all $l > 0$ and for all $1 \leq j \leq L$.*

Proof. When $l = 1$, this is clear from the definition $\mathcal{E}_{0,j} = 0$ and the fact $H(x \otimes y) = 0$ or 1 for any $x \in B_1$.

Now we investigate possible values for $\mathcal{E}_{l,j} - \mathcal{E}_{l-1,j}$. We show that the difference between tableaux for $u_l^{(j)}$ and $u_{l-1}^{(j)}$ is only one letter, namely, if $u_{l-1}^{(j)} = (x_1, x_2)$, then $u_l^{(j)} = (x_1+1, x_2)$ or $u_l^{(j)} = (x_1, x_2+1)$. We show the claim by induction on j. For $j = 0$, it is true because $u_{l-1}^{(0)} = u_{l-1} = (l-1, 0)$ and $u_l^{(0)} = u_l = (l, 0)$. Suppose that the above claim holds for all $j < k$ for some k. In order to compare $u_{l-1}^{(k)}$ and $u_l^{(k)}$, consider the relations $u_{l-1}^{(k-1)} \otimes b_k \simeq b'_{l-1,k} \otimes u_{l-1}^{(k)}$ and $u_l^{(k-1)} \otimes b_k \simeq b'_{l,k} \otimes u_l^{(k)}$. By the assumption, the difference between $u_{l-1}^{(k-1)}$ and $u_l^{(k-1)}$ is one letter. Recall that in calculating the combinatorial R by the graphical rule (section A.2), order of making pairs is arbitrary. Therefore, in $u_l^{(k-1)} \otimes b_k$, first we can make all pairs that appear in $u_{l-1}^{(k-1)} \otimes b_k$, and then make the remaining one pair. This means the difference of number of unwinding pairs, i.e., $\mathcal{E}_{l,k} - \mathcal{E}_{l-1,k}$ is 0 or 1. To make the induction proceed, note that this fact means the difference between $u_{l-1}^{(k)}$ and $u_l^{(k)}$ is also one letter. $\square$

Let $b = b_1 \otimes \cdots \otimes b_L \in B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_L}$ be an arbitrary (either highest or not) path. Set $N = \mathcal{E}_1(b)$. We determine the pair of numbers (μ_1, r_1), (μ_2, r_2), $\cdots$, (μ_N, r_N) by Step (i)–(iv).

(i) Draw a table containing $(\mathcal{E}_{l,j} - \mathcal{E}_{l-1,j} = 0, 1)$ at the position (l, j), i.e., at the l th row and the j th column. We call this table local energy distribution.
(ii) Starting from the rightmost 1 in the $l = 1$ st row, pick one 1 from each successive row. The one in the $(l+1)$ th row must be weakly right of the one selected in the l th row. If there is no such 1 in the $(l+1)$ th row, the position of the lastly picked 1 is called (μ_1, j_1). Change all the selected 1 into 0.
(iii) Repeat Step (ii) $(N-1)$ times to further determine (μ_2, j_2), $\cdots$, (μ_N, j_N) thereby making all 1 into 0.
(iv) Determine $r_1, \cdots, r_N$ by

$$r_k = \sum_{i=1}^{j_k - 1} \min(\mu_k, \lambda_i) + \mathcal{E}_{\mu_k, j_k} - 2\sum_{i=1}^{j_k} \mathcal{E}_{\mu_k, i}. \tag{2.2}$$

One may replace the procedure (ii) by

(ii)' Starting from any one of the lowest 1, pick one 1 from each preceding row. The one in the $(l-1)$th row must be weakly left and nearest of the one selected in the lth row. The position of the firstly picked 1 is called (μ_1, j_1). Change all the selected 1 into 0.

Our main result in this section is the following theorem, which gives a crystal theoretic reformulation of the KKR map ϕ.

Theorem 2.2. *The above procedure (i)–(iv) is well defined and $(\lambda, (\mu, r))$ coincides with the (unrestricted) rigged configuration $\phi(b)$. The procedure (i), (ii)',(ii),(iv) is also well defined and leads to the same rigged configuration up to a permutation of (μ_k, j_k)'s.*

The proof will be given in [29].

Example 2.3. Consider the path which will also be treated in Example B.5:

$$b = \boxed{1111} \otimes \boxed{11} \otimes \boxed{22} \otimes \boxed{12} \otimes \boxed{2} \otimes \boxed{122} \otimes \boxed{122} \otimes \boxed{1112}$$

According to Step (i), the local energy distribution is given in the following table (j stands for column coordinate of the table).

	1111	11	22	12	2	122	122	1112
$\mathcal{E}_{1,j}-\mathcal{E}_{0,j}$	0	0	1	0	1	0	1	0
$\mathcal{E}_{2,j}-\mathcal{E}_{1,j}$	0	0	1	0	0	0	0	1
$\mathcal{E}_{3,j}-\mathcal{E}_{2,j}$	0	0	0	1	0	0	0	0
$\mathcal{E}_{4,j}-\mathcal{E}_{3,j}$	0	0	0	0	0	1	0	0
$\mathcal{E}_{5,j}-\mathcal{E}_{4,j}$	0	0	0	0	0	1	0	0
$\mathcal{E}_{6,j}-\mathcal{E}_{5,j}$	0	0	0	0	0	0	1	0
$\mathcal{E}_{7,j}-\mathcal{E}_{6,j}$	0	0	0	0	0	0	0	0

Following Step (ii) and Step (iii), letters 1 contained in the above table are classified into 3 groups, as indicated in the following table.

	1111	11	22	12	2	122	122	1112
$\mathcal{E}_{1,j}-\mathcal{E}_{0,j}$			3		2^*		1	
$\mathcal{E}_{2,j}-\mathcal{E}_{1,j}$			3					1^*
$\mathcal{E}_{3,j}-\mathcal{E}_{2,j}$				3				
$\mathcal{E}_{4,j}-\mathcal{E}_{3,j}$						3		
$\mathcal{E}_{5,j}-\mathcal{E}_{4,j}$						3		
$\mathcal{E}_{6,j}-\mathcal{E}_{5,j}$							3^*	
$\mathcal{E}_{7,j}-\mathcal{E}_{6,j}$								

The cardinalities of the 3 groups are 2, 1 and 6, respectively. From the positions marked with $*$, we find $(\mu_1, j_1) = (2, 8)$, $(\mu_2, j_2) = (1, 5)$ and $(\mu_3, j_3) = (6, 7)$. Now we evaluate riggings r_i according to the rule (2.2).

$$
\begin{aligned}
r_1 &= \sum_{i=1}^{8-1} \min(2, \lambda_i) + \mathcal{E}_{2,8} - 2\sum_{i=1}^{8} \mathcal{E}_{2,i} \\
&= (2+2+2+2+1+2+2) + 1 - 2(0+0+2+0+1+0+1+1) \\
&= 4, \\
r_2 &= \sum_{i=1}^{5-1} \min(1, \lambda_i) + \mathcal{E}_{1,5} - 2\sum_{i=1}^{5} \mathcal{E}_{1,i} \\
&= (1+1+1+1) + 1 - 2(0+0+1+0+1) \\
&= 1, \\
r_3 &= \sum_{i=1}^{7-1} \min(6, \lambda_i) + \mathcal{E}_{6,7} - 2\sum_{i=1}^{7} \mathcal{E}_{6,i} \\
&= (4+2+2+2+1+3) + 2 - 2(0+0+2+1+1+2+2) \\
&= 0.
\end{aligned}
$$

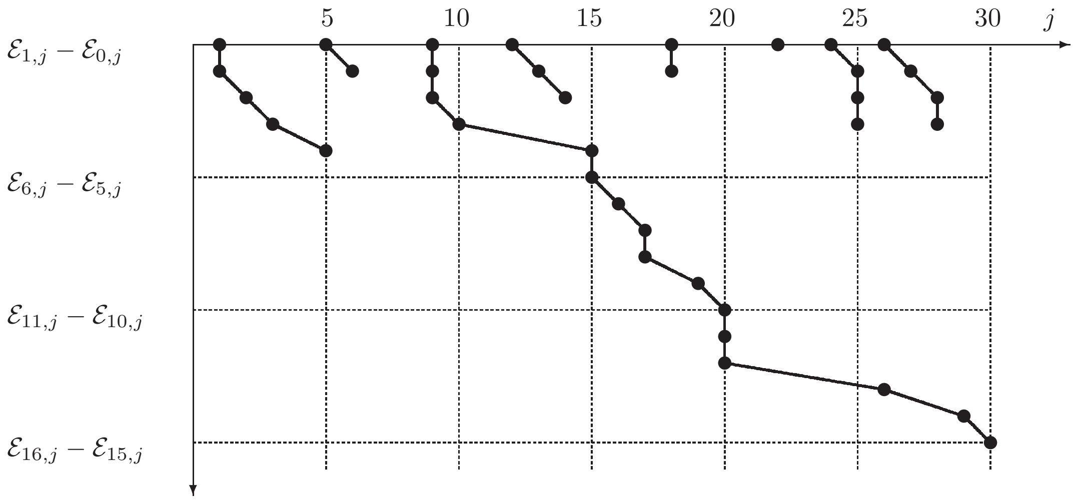
Therefore we obtain $(\mu_1, r_1) = (2, 4)$, $(\mu_2, r_2) = (1, 1)$ and $(\mu_3, r_3) = (6, 0)$, which coincide with the calculation in Example B.5.

The reader should compare the above local energy distribution and box adding procedure fully exhibited in Example B.5. Then it will be observed that the complicated combinatorial procedure in Definition B.4 is reduced to rather automatic applications of the combinatorial R and energy functions.

Example 2.4. Theorem 2.2 provides a panoramic view on the combinatorial procedure of the KKR bijection from energy distribution. To show a typical example, we pick the following long path (length 30).

$$
\begin{aligned}
&\boxed{22}\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{1}\otimes\boxed{1122}\otimes\boxed{112}\otimes\boxed{1}\otimes\boxed{11}\otimes\boxed{222}\otimes\boxed{12}\otimes\boxed{11}\otimes\boxed{2} \\
&\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{22}\otimes\boxed{2}\otimes\boxed{1122}\otimes\boxed{22}\otimes\boxed{2}\otimes\boxed{222}\otimes\boxed{1}\otimes\boxed{112}\otimes\boxed{1}\otimes\boxed{12} \\
&\otimes\boxed{1222}\otimes\boxed{11122}\otimes\boxed{2}\otimes\boxed{22}\otimes\boxed{2}\otimes\boxed{2}
\end{aligned}
$$

Then, the local energy distribution takes the following form.



In the above table, letters 1 in the local energy distribution are represented by "$\bullet$", and letters 0 are suppressed. According to Step (ii) and Step (iii), $\bullet$ belonging to the same group are joined by thick lines. We see there are 8 groups whose cardinalities are 5, 2, 16, 3, 2, 1, 4, 4 from left to right, respectively.

By using the formula (2.2), we get the unrestricted rigged configuration as follows: $(\mu_1, r_1) = (4, 8)$, $(\mu_2, r_2) = (4, 8)$, $(\mu_3, r_3) = (1, 10)$, $(\mu_4, r_4) = (2, 8)$, $(\mu_5, r_5) = (3, 2)$, $(\mu_6, r_6) = (16, -15)$, $(\mu_7, r_7) = (2, 0)$, $(\mu_8, r_8) = (5, -5)$. The vacancy numbers for each row is $p_{16} = -15$, $p_5 = 7$, $p_4 = 10$, $p_3 = 14$, $p_2 = 16$ and $p_1 = 14$. Note that since the path in this example is not highest, the resulting unrestricted rigged configuration has negative riggings and vacancy numbers.

3. Generalized periodic box-ball system

Here we extend the inverse scattering formalism [21] of the simplest periodic box-ball system [40, 39] to general higher spins. The relevant time evolutions and associated energy will be denoted by T_l and E_l for distinction from $\mathcal{T}_l$ and $\mathcal{E}_l$ for the non-periodic case. Most of the proofs will be omitted as they are similar (but somewhat more involved) to [21]. Our new algorithm for the KKR map (Theorem 2.2), adapted to the periodic boundary condition, serves as a simple algorithm for the direct scattering transform.

3.1. **Time evolution.** Fix the integer $L, s \in \mathbb{Z}_{\geq 1}$ throughout. Set

$$\mathcal{P} = B_s^{\otimes L}. \tag{3.1}$$

We will also write $\mathrm{Aff}(\mathcal{P}) = \mathrm{Aff}(B_s)^{\otimes L}$. An element of $\mathcal{P}$ is called a path. A path b is *highest* if $\tilde{e}_1 b = 0$. The weight of a path $b = b_1 \otimes \cdots \otimes b_L$ is given by $\mathrm{wt}(b) = \mathrm{wt}(b_1) + \cdots + \mathrm{wt}(b_L)$. We write $\mathrm{wt}(b) > 0$ ($\mathrm{wt}(b) < 0$) when it belongs to $\mathbb{Z}_{>0}\Lambda_1$ ($\mathbb{Z}_{<0}\Lambda_1$).

Our generalized periodic box-ball system is a dynamical system on a subset of $\mathcal{P}$ equipped with the commuting family of time evolutions $T_1, T_2, \ldots$. Let $b = b_1 \otimes \cdots \otimes b_L \in \mathcal{P}$ be a path and $l \in \mathbb{Z}_{\geq 1}$. For $v_l \in B_l$, suppose

$$\zeta^0 v_l \otimes (\zeta^0 b_1 \otimes \cdots \otimes \zeta^0 b_L) \simeq (\zeta^{-d_1} \tilde{b}_1 \otimes \cdots \otimes \zeta^{-d_L} \tilde{b}_L) \otimes \zeta^e v'_l \tag{3.2}$$

holds under the isomorphism $\mathrm{Aff}(B_l) \otimes \mathrm{Aff}(\mathcal{P}) \simeq \mathrm{Aff}(\mathcal{P}) \otimes \mathrm{Aff}(B_l)$, where the right hand side is unambiguously determined from the left hand side. ($e = d_1 + \cdots + d_L$.) We say that b is T_l-*evolvable* if the following (i) existence and (ii) uniqueness are satisfied:

(i) there exists $v_l \in B_l$ such that $v'_l = v_l$.
(ii) if there are more than one such v_l, $\tilde{b}_1 \otimes \cdots \otimes \tilde{b}_L$ is independent of their choice.

If b is T_l-evolvable, we define $T_l(b) = \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_L (\in \mathcal{P})$. Otherwise we set $T_l(b) = 0$. In this sense, we will also write $T_l(b) \neq 0$ to mean that b is T_l-evolvable.

Lemma 3.1. *If $b = b_1 \otimes \cdots \otimes b_L$ is T_l-evolvable, not only $\tilde{b}_1 \otimes \cdots \otimes \tilde{b}_L$ but also $d_1, \ldots, d_L$ and e in (3.2) are independent of the possibly nonunique choices of $v_l = v'_l$.*

Thanks to this lemma we are entitled to define $E_l(b) = e (\in \mathbb{Z}_{\geq 0})$ by (3.2) for a T_l-evolvable path b. Actually v_l can be nonunique only if $l > s$ and $\mathrm{wt}(p) = 0$. The operations $T_1, T_2, \ldots$ form a family of time evolution operators associated with the energy $E_1, E_2, \ldots$. These definitions can be summarized in

$$\zeta^0 v_l \otimes b \simeq T_l(b) \otimes \zeta^{E_l(b)} v_l \tag{3.3}$$

up to the spectral parameter for T_l-evolvable b. Pictorially, (3.2) looks as

$$v_l = v(0) \underset{\tilde{b}_1}{\overset{b_1}{+}} v(1) \underset{\tilde{b}_2}{\overset{b_2}{+}} v(2) \;\cdots\; \underset{\tilde{b}_{L-1}}{\overset{b_{L-1}}{+}} v(L-1) \underset{\tilde{b}_L}{\overset{b_L}{+}} v(L) = v'_l \tag{3.4}$$

Clearly the time evolutions are weight preserving, i.e., $\mathrm{wt}(T_l(b)) = \mathrm{wt}(b)$ when $T_l(b) \neq 0$.

Since the combinatorial R is trivial on $B_s \otimes B_s$ (see (A.7)), we have the unique choice $v_s = v'_s = b_L$ in (3.2), saying that a path is always T_s-evolvable and T_s acts as a cyclic shift:

$$T_s(b_1 \otimes b_2 \otimes \cdots \otimes b_L) = b_L \otimes b_1 \otimes \cdots \otimes b_{L-1}. \tag{3.5}$$

If $s = 1$, all the paths are T_l-evolvable for any $l \geq 1$ [21]. However this is no longer the case for $s > 1$. A similar situation is known also in the higher rank extensions [22]. Here we treat such a subtlety characteristic in the periodic setting.

We simply say that $b \in \mathcal{P}$ is *evolvable* if it is T_l-evolvable for all $l \in \mathbb{Z}_{\geq 1}$. We warn that "$b$ is T_l-evolvable" is different from "$T_l(b)$ is evolvable". The former means $T_l(b) \neq 0$ whereas the latter does $T_k T_l(b) \neq 0$ for all $k \geq 1$. Here is a characterization of evolvable paths.

Proposition 3.2. *A path $b = b_1 \otimes \cdots \otimes b_L$ is evolvable if and only if $b_i = \boxed{1\ldots1}$ or $b_i = \boxed{2\ldots2}$ for some i.*

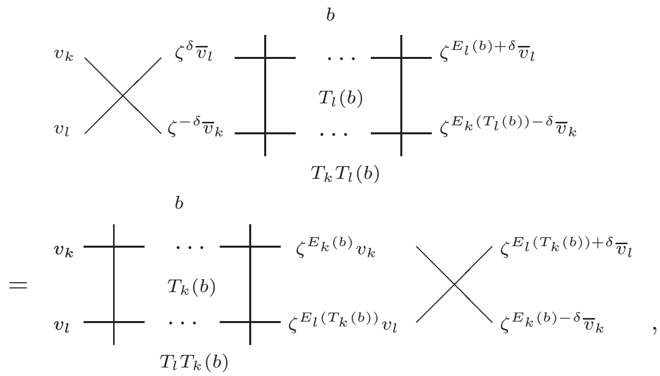
The proof of the proposition also tells the way to construct v_l that makes (3.3) hold for a given path b. For $l \geq s$, determine $v_l \in B_l$ by (see (A.6) for u_l)

$$\begin{aligned} u_l \otimes b &\simeq b' \otimes v_l \quad && \text{if } \mathrm{wt}(b) \geq 0, \\ \omega(u_l) \otimes b &\simeq b' \otimes v_l \quad && \text{if } \mathrm{wt}(b) < 0, \end{aligned} \tag{3.6}$$

where $b' \in \mathcal{P}$ is another path. So obtained v_l is shown to satisfy $v_l \otimes b \simeq T_l(b) \otimes v_l$ under $B_l \otimes \mathcal{P} \simeq \mathcal{P} \otimes B_l$. One may either use the latter relation in (3.6) to define v_l when $\mathrm{wt}(b) = 0$. For $l < s$, one has the unique v_l from $v \otimes b \simeq b' \otimes v_l$ for arbitrary $v \in B_l$ if b is evolvable. Then $v_l \otimes b \simeq T_l(b) \otimes v_l$ is again valid.

Theorem 3.3. *Suppose $b, T_l(b)$ and $T_k(b)$ are evolvable. Then the commutativity $T_l T_k(b) = T_k T_l(b)$ and the energy conservation $E_l(T_k(b)) = E_l(b), E_k(T_l(b)) = E_k(b)$ hold.*

Proof. Take v_k for b and v_l for $T_k(b)$ as in (3.6). Set $R(\zeta^0 v_l \otimes \zeta^0 v_k) = \zeta^{-\delta} \overline{v}_k \otimes \zeta^{\delta} \overline{v}_l$ and regard b as an element of $\mathrm{Aff}(\mathcal{P})$. By using the combinatorial R, one can reorder $\zeta^0 v_l \otimes \zeta^0 v_k \otimes b$ in two ways along the isomorphism $\mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_k) \otimes \mathrm{Aff}(\mathcal{P}) \simeq \mathrm{Aff}(\mathcal{P}) \otimes \mathrm{Aff}(B_k) \otimes \mathrm{Aff}(B_l)$ as follows:



where the equality is due to the Yang-Baxter equation. The outputs have been identified with $T_kT_l(b), \zeta^{E_k(T_l(b))-\delta}\overline{v}_k$, etc. In particular the uniqueness (ii) stated under (3.2) guarantees that $\overline{v}_k \otimes T_l(b) \simeq T_kT_l(b) \otimes \overline{v}_k$ and $\overline{v}_l \otimes b \simeq T_l(b) \otimes \overline{v}_l$ up to the spectral parameter. The sought relations $T_lT_k(b) = T_kT_l(b)$ and $E_l(T_k(b)) = E_l(b), E_k(T_l(b)) = E_k(b)$ are obtained by comparing the two sides. $\square$

Let s_0, s_1 be the Weyl group operators (A.3) and ω be the involution (A.4) acting on the crystal $\mathcal{P}$. Then $\widehat{W}(A_1^{(1)}) = \langle \omega, s_0, s_1 \rangle$ forms the extended affine Weyl group of type $A_1^{(1)}$. The time evolutions T_l and the energy E_l enjoy the symmetry under $\widehat{W}(A_1^{(1)})$.

Proposition 3.4. *Let b be an evolvable path. Then for any $w \in \widehat{W}(A_1^{(1)})$, $w(b)$ is also evolvable and the commutativity $wT_l(b) = T_l(w(b))$ and the invariance $E_l(w(b)) = E_l(b)$ are valid.*

In particular, the relation

$$T_l = \omega \circ T_l \circ \omega \tag{3.7}$$

exchanging the letters $1 \leftrightarrow 2$ is useful.

Any path is T_l-evolvable for $l \geq s$. In fact, for l sufficiently large the time evolution T_l and the energy E_l admit a simple description as follows.

Proposition 3.5. *For any path $b \in \mathcal{P}$, there exists $k \geq s$ such that $T_l(b)$ and $E_l(b)$ are independent of l for $l \geq k$. Denoting them by $T_\infty(b)$ and $E_\infty(b)$, one has*

$$\begin{aligned} T_\infty(b) &= \omega(s_0(b)), \quad \mathrm{wt}(b) = p_\infty \Lambda_1 \quad \textit{if } \mathrm{wt}(b) \geq 0, \\ T_\infty(b) &= \omega(s_1(b)), \quad \mathrm{wt}(b) = -p_\infty \Lambda_1 \quad \textit{if } \mathrm{wt}(b) \leq 0, \end{aligned}$$

where $p_\infty = Ls - 2E_\infty(b)$ according to (3.10). In particular, $T_\infty(b) = \omega(b)$ if $\mathrm{wt}(b) = 0$.

Example 3.6. For $b = \boxed{112} \otimes \boxed{111} \otimes \boxed{222} \otimes \boxed{122} \otimes \boxed{112}$ having a positive weight, we have

$$\begin{aligned} T_1(b) &= \boxed{122} \otimes \boxed{111} \otimes \boxed{122} \otimes \boxed{222} \otimes \boxed{111}, \\ T_2(b) &= \boxed{112} \otimes \boxed{112} \otimes \boxed{112} \otimes \boxed{222} \otimes \boxed{112}, \\ T_3(b) &= \boxed{112} \otimes \boxed{112} \otimes \boxed{111} \otimes \boxed{222} \otimes \boxed{122}, \\ T_4(b) &= \boxed{122} \otimes \boxed{112} \otimes \boxed{111} \otimes \boxed{122} \otimes \boxed{122}, \\ T_l(b) &= \boxed{122} \otimes \boxed{122} \otimes \boxed{111} \otimes \boxed{112} \otimes \boxed{122} \quad (l \geq 5). \end{aligned}$$

So $T_\infty(b) = T_5(b)$. On the other hand, 0-signature and reduced 0-signature of b read

$$\begin{array}{ccccccccc} \boxed{112} & \otimes & \boxed{111} & \otimes & \boxed{222} & \otimes & \boxed{122} & \otimes & \boxed{112} \\ --+ & & --- & & +++ & & -++ & & --+ \\ -- & & -- & & ++ & & & & + \end{array}$$

Thus $s_0(b) = \boxed{112} \otimes \tilde{e}_0 \boxed{111} \otimes \boxed{222} \otimes \boxed{122} \otimes \boxed{112}$, which coincides with $\omega(T_\infty(b))$.

For an evolvable path $b \in \mathcal{P}$, we have the time evolution $T_l(b) \in \mathcal{P}$ and the associated energy $E_l(b) \in \mathbb{Z}_{\geq 0}$ for all $l \geq 1$. This leads us to introduce the "iso-level" set

$$\widehat{\mathcal{P}}(m) = \{b \in \mathcal{P} \mid b : \text{evolvable},\ E_l(b) = \sum_{k \geq 1} \min(l,k) m_k\} \tag{3.8}$$

labeled with the sequence $m = \{m_k \mid k \geq 1\}$. We shall always take it for granted that $\{m_k\}$ and $\{E_l\}$ are in one-to-one correspondence via

$$E_l = \sum_{k \geq 1} \min(l,k) m_k, \quad m_k = -E_{k-1} + 2E_k - E_{k+1}\ (E_0 = 0). \tag{3.9}$$

We also use the vacancy number

$$p_j = L \min(s,j) - 2E_j. \tag{3.10}$$

The following result is due to T. Takagi.

Proposition 3.7 ([32])**.** *For any path $b \in \widehat{\mathcal{P}}(m)$ with $\mathrm{wt}(b) \geq 0$, its time evolution $\big(\prod_l T_l^{d_l}\big)(b)$ becomes highest under appropriate choices of $\{d_l\}$* [2].

Such $\{d_l\}$ is not unique. Cyclic shift $T_s^{d_s}$ is not enough to achieve this in general. From Proposition 3.7 one can show

Proposition 3.8. *$\widehat{\mathcal{P}}(m) \neq \emptyset$ if and only if $\forall p_j \geq 0$.*

Henceforth we assume $\forall p_j \geq 0$. If b belongs to $\widehat{\mathcal{P}}(m)$ and $T_l(b)$ is evolvable, then $T_l(b) \in \widehat{\mathcal{P}}(m)$ must hold because of $E_k(T_l(b)) = E_k(b)$ by Theorem 3.3. However, the point here is that even if a path b is evolvable, it is *not* guaranteed in general that its time evolution $T_l(b)$ is again evolvable.

Example 3.9. $b_1 = \boxed{11} \otimes \boxed{22}$ and $b_2 = \boxed{22} \otimes \boxed{11}$ are evolvable, but $T_1(b_1) = T_1(b_2) = \boxed{12} \otimes \boxed{12}$ is not. See Proposition 3.2. The situation is depicted as

$$\begin{tikzcd}
\boxed{11} \otimes \boxed{22} \arrow[dr, "T_1"] & & \\
& \boxed{12} \otimes \boxed{12} \arrow[r, "T_1"] & 0 \\
\boxed{22} \otimes \boxed{11} \arrow[ur, "T_1"'] & &
\end{tikzcd}$$

Thus the set $T_l(\widehat{\mathcal{P}}(m))$ can contain non-evolvable paths in general. On the other hand, all the evolvable paths in $T_l(\widehat{\mathcal{P}}(m))$ must share the same energy spectrum $\{E_l\}$ as $\widehat{\mathcal{P}}(m)$ by virtue of Theorem 3.3. Therefore, what holds in general is

$$T_l(\widehat{\mathcal{P}}(m)) = (\text{subset of } \widehat{\mathcal{P}}(m)) \sqcup \{T_l(b) : \text{non-evolvable} \mid b \in \widehat{\mathcal{P}}(m)\}.$$

A natural question is to find a pleasant situation where T_l acts on $\widehat{\mathcal{P}}(m)$ as a bijection. This is answered in

Proposition 3.10. *$T_l(\widehat{\mathcal{P}}(m)) = \widehat{\mathcal{P}}(m)$ holds for all l if and only if $(E_1, E_2) \neq (L/2, L)$.*

So this is always satisfied if L is odd. For an evolvable path b with even length L, the condition $(E_1, E_2) = (L/2, L)$ is equivalent to

$$b = 11(c_1) \otimes (c_2)22 \otimes \cdots \otimes (c_L)22 \quad \text{or} \quad b = (c_1)22 \otimes 11(c_2) \otimes \cdots \otimes 11(c_L)$$

for some $c_i \in B_{s-2}$, where 11 and 22 alternate. Here for example, $11(c) = \boxed{11122}$ and $(c)22 = \boxed{12222}$ for $c = \boxed{122} \in B_3$. (Thus such b can exist only for $s \geq 2$.) The two paths

[2] Actually T_s and T_{s-1} have been shown to suffice.

in Example 3.9 correspond to the case $(E_1, E_2) = (1, 2)$ with $L = 2$. For $\widehat{\mathcal{P}}(m)$ such that $(E_1, E_2) \neq (L/2, L)$, the inverse time evolution is given by

$$T_l^{-1} = \varrho \circ T_l \circ \varrho, \tag{3.11}$$

where ϱ is defined by

$$\varrho(b_1 \otimes b_2 \otimes \cdots \otimes b_L) = b_L \otimes \cdots \otimes b_2 \otimes b_1.$$

To summarize Theorem 3.3, Proposition 3.4 and Proposition 3.10, each set $\widehat{\mathcal{P}}(m)$ of evolvable paths is characterized by the conserved quantity $E_l = \sum_{k \geq 1} \min_k(l, k) m_k$ called energy, and enjoys the invariance under

(i) the extended affine Weyl group $\widehat{W}(A_1^{(1)})$,
(ii) the commuting family of invertible time evolutions $\{T_l \mid l \geq 1\}$.

$\widehat{\mathcal{P}}(m)$ is non-empty if $\forall p_j \geq 0$. The invariance (ii) is valid if $(E_1, E_2) \neq (L/2, L)$ is further satisfied.

3.2. **Action-angle variable.** From now on, we assume that $m = \{m_j\}$ satisfies

$$\forall p_j \geq 1. \tag{3.12}$$

See (3.9) and (3.10). This fulfills the conditions in Propositions 3.8 and 3.10 since $p_1 = L - 2E_1$. The set $\widehat{\mathcal{P}}(m)$ is decomposed into a disjoint union of fixed weight subsets:

$$\begin{aligned} \widehat{\mathcal{P}}(m) &= \mathcal{P}(m) \sqcup \omega(\mathcal{P}(m)), \\ \mathcal{P}(m) &= \{b \in \widehat{\mathcal{P}}(m) \mid \mathrm{wt}(b) = p_\infty \Lambda_1\}. \end{aligned} \tag{3.13}$$

In view of (3.7), dynamics on $\widehat{\mathcal{P}}(m)$ is reduced to the commuting family of invertible time evolutions $\{T_l\}$ on the fixed (positive) weight subset $\mathcal{P}(m)$:

$$T_l \,:\, \mathcal{P}(m) \to \mathcal{P}(m). \tag{3.14}$$

We present the inverse scattering transform that linearizes the dynamics (3.14) and an explicit solution of the initial value problem. For a general background on the inverse scattering method, see [1, 8]. In our approach the direct scattering transform is formulated either by a modified KKR bijection as in the $s = 1$ case [21] or by an appropriate extension of the procedure in Theorem 2.2 to a periodic setting.

First we introduce the action-angle variables. For the paths belonging to $\mathcal{P}(m)$, the action variable is just the conserved quantity $m = \{m_j\}$ or equivalently $\{E_l\}$ (3.9). It may also be presented as the Young diagram μ having m_j rows with length j. Let $H = \{j_1 < \cdots < j_g\}$ be the set of distinct row lengths of μ, namely, $j \in H \leftrightarrow m_j > 0$. The set $\mathcal{J}(m)$ of angle variables is defined by

$$\begin{aligned} \mathcal{J}(m) &= \Big((\mathbb{Z}^{m_{j_1}} \times \cdots \times \mathbb{Z}^{m_{j_g}})/\Gamma \; - \Delta \Big)_{\mathrm{sym}}, \\ \Gamma &= A(\mathbb{Z}^{m_{j_1}} \times \cdots \times \mathbb{Z}^{m_{j_g}}). \end{aligned} \tag{3.15}$$

Here $A = (A_{j\alpha,k\beta})$ is the matrix of size $m_{j_1} + \cdots + m_{j_g}$ having a block structure:

$$A_{j\alpha,k\beta} = \delta_{j,k}\delta_{\alpha,\beta}(p_j + m_j) + 2\min(j,k) - \delta_{j,k}, \tag{3.16}$$

where $j, k \in H$ and $1 \leq \alpha \leq m_j, 1 \leq \beta \leq m_k$. A is symmetric and positive definite under the assumption (3.12) [19]. Δ is the subset of $(\mathbb{Z}^{m_{j_1}} \times \cdots \times \mathbb{Z}^{m_{j_g}})/\Gamma$ having coincident components within a block:

$$\Delta = \{(I_{j,\alpha})_{j \in H, 1 \leq \alpha \leq m_j} \mid I_{j,\alpha} = I_{j,\beta} \text{ for some } j \in H, 1 \leq \alpha \neq \beta \leq m_j\}. \tag{3.17}$$

In (3.15), $-\Delta$ signifies the complement of Δ. The subscript sym means the identification under the exchange of components within blocks via the symmetric group $\mathfrak{S}_{m_{j_1}} \times \cdots \times \mathfrak{S}_{m_{j_g}}$. We introduce the time evolution of the angle variables by

$$\begin{aligned} T_l \,:\, \mathcal{J}(m) &\longrightarrow \mathcal{J}(m), \\ (I_{j,\alpha}) &\longmapsto (I_{j,\alpha} + \min(l,j)), \end{aligned}$$

which makes sense because it obviously preserves the set $(\mathbb{Z}^{m_{j_1}} \times \cdots \times \mathbb{Z}^{m_{j_g}})/\Gamma \; - \Delta$. We shall simply write this as

$$T_l(\mathbf{I}) = \mathbf{I} + \mathbf{h}_l. \tag{3.18}$$

Namely, $\mathbf{h}_l = (\min(j,l))_{j \in H, 1 \le \alpha \le m_j} \in \mathbb{Z}^{m_{j_1} + \cdots + m_{j_g}}$ is the velocity of the angle variable $\mathbf{I} = (I_{j,\alpha})$ under the time evolution T_l.

3.3. **Direct scattering.** We introduce the direct scattering map $\Phi : \mathcal{P}(m) \to \mathcal{J}(m)$. A quick formulation is due to a modified KKR bijection as done in [21] for $s = 1$. Note that Proposition 3.7 tells that $\mathcal{P}(m)$ is the $\{T_l\}$-orbit of highest paths having the KKR configuration $m = \{m_j\}$. Thus we express a given $b \in \mathcal{P}(m)$ as $b = \left(\prod_l T_l^{d_l}\right)(b_+)$ in terms of a highest path $b_+ \in \mathcal{P}(m)$. Let (μ, r) be the rigged configuration for b_+, where the appearance of the μ corresponding to $m = \{m_j\}$ is due to Theorem 3.3. Let the rigging attached to the length $j (\in H)$ rows of μ be $0 \le r_{j,1} \le \cdots \le r_{j,m_j} \le p_j$. Consider the element

$$\mathbf{I} + \sum_l d_l \mathbf{h}_l \mod \Gamma \quad \in \mathcal{J}(m), \quad \text{where } \mathbf{I} = (r_{j,\alpha} + \alpha - 1)_{j \in H, 1 \le \alpha \le m_j}. \tag{3.19}$$

$r_{j,\alpha} + \alpha - 1$ is strictly increasing with α, therefore $\mathbf{I} + \sum_l d_l \mathbf{h}_l \mod \Gamma$ belongs to $(\mathbb{Z}^{m_{j_1}} \times \cdots \times \mathbb{Z}^{m_{j_g}})/\Gamma - \Delta$. Given $b \in \mathcal{P}(m)$, the choice of $\{d_l\}$ and the highest path b_+ such that $b = \left(\prod_l T_l^{d_l}\right)(b_+)$ is not unique in general. This non-uniqueness is to be cancelled by $\mathrm{mod}\,\Gamma$. In fact we have

Theorem 3.11. *The rule* $\Phi(b) = \mathbf{I} + \sum_l d_l \mathbf{h}_l \bmod \Gamma$ *specified by (3.19) is a bijection* $\Phi : \mathcal{P}(m) \to \mathcal{J}(m)$*, and the following commutative diagram is valid:*

$$\begin{tikzcd} \mathcal{P}(m) \arrow[r, "\Phi"] \arrow[d, "T_l"'] & \mathcal{J}(m) \arrow[d, "T_l"] \\ \mathcal{P}(m) \arrow[r, "\Phi"] & \mathcal{J}(m) \end{tikzcd} \tag{3.20}$$

An alternative way to define the direct scattering map Φ is obtained by a periodic extension of the procedure (i), (ii)', (iii), (iv) in Theorem 2.2. This option is valid under a certain condition which we shall explain after (3.24). It is more direct than the above one in that the relation $b = \left(\prod_l T_l^{d_l}\right)(b_+)$ need not be found. Here we illustrate it along the example:

$$b = \boxed{122} \otimes \boxed{122} \otimes \boxed{112} \otimes \boxed{112} \otimes \boxed{111} \otimes \boxed{122} \otimes \boxed{111} \otimes \boxed{111} \otimes \boxed{112} \in B_3^{\otimes 9}. \tag{3.21}$$

This path has appeared as the $t = 0$ case in the introduction. Let $v_l \in B_l$ be the element satisfying $v_l \otimes b \simeq T_l(b) \otimes v_l$. It is unique under the condition (3.12) and can be found by (3.6). The energy is given by $E_1 = 4, E_2 = 7, E_3 = 8, E_l = 9\,(l \ge 4)$. So the action variable is

$$\mu = \ydiagram{4,2,2,1}$$

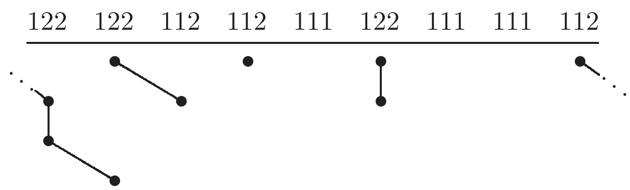
Local energy $E_{l,k} = H(v(k-1) \otimes b_k)$ is determined by using $v(k)$ in (3.4). The distribution of $\delta E_{l,k} = E_{l,k} - E_{l-1,k}$ looks as

	122	122	112	112	111	122	111	111	112
$\delta E_{1,k}$	0	1	0	1	0	1	0	0	1
$\delta E_{2,k}$	1	0	1	0	0	1	0	0	0
$\delta E_{3,k}$	1	0	0	0	0	0	0	0	0
$\delta E_{4,k}$	0	1	0	0	0	0	0	0	0

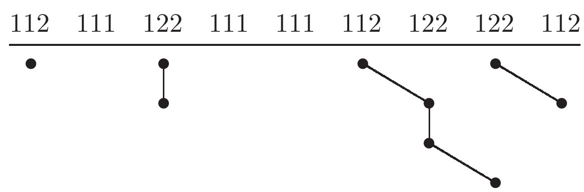
We group 1's by a periodic analogue of the procedure (i), (ii)', (iii), (iv) in Theorem 2.2. Pick a lowest 1, say $\delta E_{l,k} = 1$ at the lth row. If there are more than one such k, any choice is possible. Let the rightmost 1 in

$$\delta E_{l-1,k+1}, \ldots, \delta E_{l-1,L-1}, \delta E_{l-1,L}, \delta E_{l-1,1}, \ldots, \delta E_{l-1,k-1}, \delta E_{l-1,k} \tag{3.22}$$

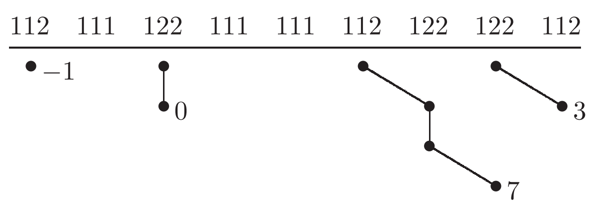
be $\delta E_{l-1,k'} = 1$. Namely, k' is the position of the rightmost 1 satisfying $k' \le k$ cyclically. Then connect $\delta E_{l,k}$ to $\delta E_{l-1,k'}$. Repeat this until the successive connection reaches some $\delta E_{1,k''}$ on the first row. This completes one group. Erase all the 1's in it and repeat the same procedure starting from a lowest 1 in the rest to form other groups until all the initial 1's are exhausted.



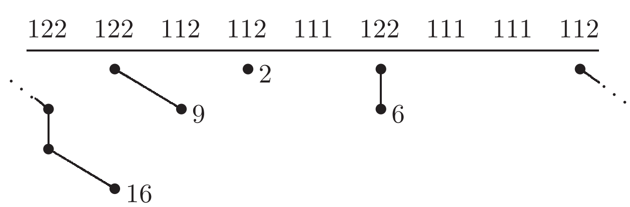
A group consisting of l dots will be called a soliton of length l. Make a cyclic shift T_s^{-d} so that all the solitons stay within the left and the right boundary. Namely, no soliton sits across the boundary. In the above, we take for example $d = 3$.



Computing the rigging of each soliton according to (2.2), we find



These values are for $T_s^{-d}(b)$. The rigging for b in question is defined to be their shift $+d\min(s, j)$ for length j solitons, leading to ($s = d = 3$ in this example)



Order the so obtained rigging for length j solitons as $r_{j,1} \le \cdots \le r_{j,m_j}$ and set

$$\mathbf{J} = (J_{j,\alpha})_{j \in H, 1 \le \alpha \le m_j}, \quad \text{where} \quad J_{j,\alpha} = r_{j,\alpha} + \alpha - 1. \tag{3.23}$$

In the present example, $H = \{1, 2, 4\}$, $(m_1, m_2, m_4) = (1, 2, 1)$, $(p_1, p_2, p_4) = (1, 4, 9)$ and

$$A = \begin{pmatrix} p_1+m_1+1 & 2\min(1,2) & 2\min(1,2) & 2\min(1,4) \\ 2\min(1,2) & p_2+m_2+3 & 3 & 2\min(2,4) \\ 2\min(1,2) & 3 & p_2+m_2+3 & 2\min(2,4) \\ 2\min(1,4) & 2\min(2,4) & 2\min(2,4) & p_4+m_4+7 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 9 & 3 & 4 \\ 2 & 3 & 9 & 4 \\ 2 & 4 & 4 & 17 \end{pmatrix},$$

so the angle variable is

$$\begin{pmatrix} J_{1,1} \\ J_{2,1} \\ J_{2,2} \\ J_{4,1} \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 9+1 \\ 16 \end{pmatrix} \quad \mathrm{mod}\ A\mathbb{Z}^4 = \mathbb{Z}\begin{pmatrix} 3 \\ 2 \\ 2 \\ 2 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 2 \\ 9 \\ 3 \\ 4 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 2 \\ 3 \\ 9 \\ 4 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 2 \\ 4 \\ 4 \\ 17 \end{pmatrix}, \tag{3.24}$$

where $+1$ is the contribution of $\alpha - 1$ in $J_{j,\alpha} = r_{j,\alpha} + \alpha - 1$.

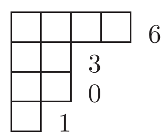
This procedure for the direct scattering map Φ works provided that there is a cyclic shift $T_s^{-d}(b)$ such that no soliton stays across the boundary. The case $s = 1$ [21] is such an example. We conjecture it for general s under the assumption (3.12).

One can show that $\mathbf{J} \bmod \Gamma$ is independent of the possible non-uniqueness of such cyclic shifts. The difference caused by such choices belong to Γ. This can be observed, for example, by comparing (3.24) and (3.25).

Let us re-derive the result (3.24) from (3.19). The latter starts, for example, from the relation $b = T_3^{-5}(b_+)$, where

$$b_+ = \boxed{111} \otimes \boxed{122} \otimes \boxed{111} \otimes \boxed{111} \otimes \boxed{112} \otimes \boxed{122} \otimes \boxed{122} \otimes \boxed{112} \otimes \boxed{112}$$

is a highest path corresponding to the rigged configuration



Thus (3.19) is evaluated as

$$\mathbf{I} - 5\mathbf{h}_3 = \begin{pmatrix} I_{1,1} \\ I_{2,1} \\ I_{2,2} \\ I_{4,1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3+1 \\ 6 \end{pmatrix} - 5\begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -10 \\ -6 \\ -9 \end{pmatrix} \quad \mathrm{mod}\ A\mathbb{Z}^4.$$

This certainly coincides with the result (3.24) since the difference

$$\begin{pmatrix} 2 \\ 6 \\ 10 \\ 16 \end{pmatrix} - \begin{pmatrix} -4 \\ -10 \\ -6 \\ -9 \end{pmatrix} = \begin{pmatrix} 6 \\ 16 \\ 16 \\ 25 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 9 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 4 \\ 17 \end{pmatrix} \tag{3.25}$$

belongs to $A\mathbb{Z}^4$.

3.4. **Inverse scattering.** According to Theorem 3.11, the dynamics of the generalized periodic box-ball system is transformed to a straight motion (3.18) in the set $\mathcal{J}(m)$ of angle variables. To complete the inverse scattering method, one needs the inverse scattering map Φ^{-1} from $\mathcal{J}(m)$ back to paths $\mathcal{P}(m)$. Under the condition (3.12), it is easy to show that any element $\mathbf{I} \in \mathcal{J}(m)$ has a (not necessarily unique) representative form $\mathbf{I} = \sum_l d_l \mathbf{h}_l + (r_{j,\alpha} + \alpha - 1)_{j \in H, 1 \le \alpha \le m_j}$ such that $0 \le r_{j,1} \le \cdots \le r_{j,m_j} \le p_j$ by using the equivalence under Γ. If μ denotes the Young diagram for $m = \{m_k\}$ and $r = (r_{j,\alpha})_{j \in H, 1 \le \alpha \le m_j}$, then (μ, r) becomes a rigged configuration. Letting b_+ be the highest path corresponding to it, $\Phi^{-1}(\mathbf{I}) := \left(\prod_l T_l^{d_l}\right)(b_+)$ is independent of the choice of the representative form and yields the inverse of Φ. Actually, one can always take $d_l = 0$ for $l \neq 1$.

Our solution of the initial value problem is achieved by the commutative diagram (3.20), namely the composition:

$$\mathcal{P}(m) \xrightarrow{\Phi} \mathcal{J}(m) \xrightarrow{\{T_l\}} \mathcal{J}(m) \xrightarrow{\Phi^{-1}} \mathcal{P}(m),$$

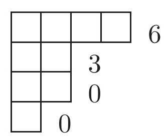
where the number of computational steps is independent of the time evolution. As an illustration we derive

$$T_2^{1000}(b) = \boxed{111} \otimes \boxed{111} \otimes \boxed{112} \otimes \boxed{112} \otimes \boxed{122} \otimes \boxed{122} \otimes \boxed{112} \otimes \boxed{111} \otimes \boxed{122},$$
$$T_4^{1000}(b) = \boxed{112} \otimes \boxed{112} \otimes \boxed{112} \otimes \boxed{111} \otimes \boxed{122} \otimes \boxed{111} \otimes \boxed{111} \otimes \boxed{122} \otimes \boxed{122}$$

for b given in (3.21). From (3.24) and (3.18), the angle variables for $T_2^{1000}(b)$ and $T_4^{1000}(b)$ are given by

$$\begin{pmatrix}1002\\2006\\2010\\2016\end{pmatrix} = \begin{pmatrix}0+82\\0+82\\3+1+82\\6+82\end{pmatrix} + 108\begin{pmatrix}3\\2\\2\\2\end{pmatrix} + 129\begin{pmatrix}2\\9\\3\\4\end{pmatrix} + 129\begin{pmatrix}2\\3\\9\\4\end{pmatrix} + 40\begin{pmatrix}2\\4\\4\\17\end{pmatrix},$$
$$\begin{pmatrix}1002\\2006\\2010\\4016\end{pmatrix} = \begin{pmatrix}0+222\\0+222\\3+1+222\\6+222\end{pmatrix} + 28\begin{pmatrix}3\\2\\2\\2\end{pmatrix} + 84\begin{pmatrix}2\\9\\3\\4\end{pmatrix} + 84\begin{pmatrix}2\\3\\9\\4\end{pmatrix} + 180\begin{pmatrix}2\\4\\4\\17\end{pmatrix}.$$

In the right hand sides, the last four terms belong to $A\mathbb{Z}^4$, hence can be neglected. The first terms correspond to T_1^{82} and T_1^{222} of the rigged configuration (+1 is removed as the "$\alpha - 1$ part")



which is mapped, under the KKR bijection, to the highest path

$$b'_+ := \boxed{111} \otimes \boxed{122} \otimes \boxed{111} \otimes \boxed{112} \otimes \boxed{111} \otimes \boxed{122} \otimes \boxed{122} \otimes \boxed{112} \otimes \boxed{112}.$$

One can check $T_1^{82}(b'_+) = T_2^{1000}(b)$ and $T_1^{222}(b'_+) = T_4^{1000}(b)$ completing the derivation.

3.5. **State counting and periodicity.** The matrix A (3.16) was originally introduced in the study of the Bethe equation at $q = 0$ [19]. From this connection we have

Theorem 3.12 ([19] Theorems 3.5, 4.9)**.**

$$|\mathcal{J}(m)| = (\det F) \prod_{j \in H} \frac{1}{m_j} \binom{p_j + m_j - 1}{m_j - 1}. \tag{3.26}$$

$$F = (F_{j,k})_{j,k \in H}, \quad F_{j,k} = \delta_{j,k} p_j + 2\min(j,k) m_k. \tag{3.27}$$

Combined with Theorem 3.11, this yields a formula for $|\mathcal{P}(m)|$, namely, the number of states characterized by the conserved quantity. For $B_3^{\otimes 9}$ and m corresponding to the Young diagram $(4, 2, 2, 1)$ considered above, we have $|\mathcal{J}(m)| = 990$.

From (3.15), all the paths $b \in \mathcal{P}(m)$ obey the relation

$$T_l^{\mathcal{N}_l}(b) = b \quad \text{if } \mathcal{N}_l \mathbf{h}_l \in \Gamma.$$

Writing $\mathcal{N}_l \mathbf{h}_l = A\mathbf{n}$, components of the vector $\mathbf{n} = (n_{j,\alpha})$ are given by

$$n_{j,\alpha} = \mathcal{N}_l \frac{\det A[j\alpha]}{\det A},$$

where $A[j\alpha]$ is obtained by replacing $(j\alpha)$th column of A by $\mathbf{h}_l$. It is elementary to check

$$\frac{\det A[j\alpha]}{\det A} = \frac{\det F[j]}{\det F}.$$

$F[j]$ is obtained from F by replacing the jth column by the l-dependent vector $(\min(l,k))_{k\in H}$. The independence on α reflects the symmetry of A (3.16) within blocks. Thus the *generic* period of $\mathcal{P}(m)$, namely the minimum $\mathcal{N}_l$ such that $\mathcal{N}_l\mathbf{h}_l \in \Gamma$ is

$$\mathcal{N}_l = \mathrm{LCM}\left(\frac{\det F}{\det F[j_1]}, \ldots, \frac{\det F}{\det F[j_g]}\right), \tag{3.28}$$

where by $\mathrm{LCM}(r_1, \ldots, r_g)$ for rational numbers $r_1, \ldots, r_g$, we mean the smallest positive integer in $\mathbb{Z}r_1 \cap \cdots \cap \mathbb{Z}r_g$. When $\det F[j_k] = 0$, the entry $\det F/\det F[j_k]$ is to be excluded. For $B_3^{\otimes 9}$ and the Young diagram $(4,2,2,1)$ in the above example, we have

$$F = \begin{pmatrix} 3 & 4 & 2 \\ 2 & 12 & 4 \\ 2 & 8 & 17 \end{pmatrix}, \ F[1] = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 12 & 4 \\ 1 & 8 & 17 \end{pmatrix}, \ F[2] = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 4 \\ 2 & 1 & 17 \end{pmatrix}, \ F[4] = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 12 & 1 \\ 2 & 8 & 1 \end{pmatrix}$$

for $l = 1$, leading to $\mathcal{N}_1 = \mathrm{LCM}\left(\frac{99}{28}, \frac{396}{13}, 99\right) = 396$. Similar calculations yield

$$\mathcal{N}_1 = 396, \quad \mathcal{N}_2 = 99, \quad \mathcal{N}_3 = 9, \quad \mathcal{N}_l = 11 \ (l \geq 4).$$

$T_l^{\mathcal{N}_l}(b) = b$ can be directly checked for b in (3.21). In fact $T_4^{11}(b) = b$ has been demonstrated in the introduction. For the fundamental period, formally the same closed formula as eq.(4.26) in [21] is valid. The formula (3.28) agrees with the most general conjecture on $A_n^{(1)}$ [22]. For $s = 1$ and $l = \infty$, it was originally obtained in [39] by a combinatorial argument.

3.6. **Ultradiscrete Riemann theta lattice.** Let us present an explicit formula for the inverse scattering map $\Phi^{-1} : \mathcal{J}(m) \to \mathcal{P}(m)$ in terms of the ultradiscrete Riemann theta function. We keep assuming the condition (3.12).

For general $\{m_j\}$, Theorem 5.1 in [24] remains valid if the vacancy number p_j is replaced by (3.10) in this paper. Here we restrict ourselves to the case $m_j = 1$ for all $j \in H = \{j_1 < \cdots < j_g\}$ for simplicity. Thus $\mathcal{J}(m)$ (3.15) and A (3.16) reduce to

$$\begin{aligned} &\mathcal{J}(m) = \mathbb{Z}^g/A\mathbb{Z}^g, \\ &A = (A_{j,k})_{j,k\in H}, \quad A_{j,k} = \delta_{j,k}p_j + 2\min(j,k). \end{aligned} \tag{3.29}$$

Following [23], we introduce the ultradiscrete Riemann theta function by

$$\begin{aligned} \Theta(\mathbf{z}) &= \lim_{\epsilon\to+0} \epsilon \log\left(\sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\Big(-\frac{{}^t\mathbf{n}A\mathbf{n}/2 + {}^t\mathbf{n}\mathbf{z}}{\epsilon}\Big)\right) \\ &= -\min_{\mathbf{n}\in\mathbb{Z}^g}\{{}^t\mathbf{n}A\mathbf{n}/2 + {}^t\mathbf{n}\mathbf{z}\}, \end{aligned} \tag{3.30}$$

which enjoys the quasi-periodicity:

$$\Theta(\mathbf{z}+\mathbf{v}) = {}^t\mathbf{v}A^{-1}(\mathbf{z}+\mathbf{v}/2) + \Theta(\mathbf{z}) \quad \text{for any} \ \ \mathbf{v} \in \Gamma = A\mathbb{Z}^g. \tag{3.31}$$

Consider the planar square lattice

$$\begin{array}{c|c|c|c|c|c}
 & B_{s_1} & B_{s_2} & B_{s_3} & & B_{s_L} \\
 & \mathbf{z}_{0,0} & \mathbf{z}_{0,1} & \mathbf{z}_{0,2} & & \mathbf{z}_{0,L} \\
\hline
B_{l_1} & & & & \cdots & \\
 & \mathbf{z}_{1,0} & \mathbf{z}_{1,1} & \mathbf{z}_{1,2} & & \mathbf{z}_{1,L} \\
\hline
B_{l_2} & & & & \cdots & \\
 & \mathbf{z}_{2,0} & \mathbf{z}_{2,1} & \mathbf{z}_{2,2} & & \mathbf{z}_{2,L} \\
 & \vdots & \vdots & \vdots & & \vdots
\end{array}$$

where we assume $s_1 = \cdots = s_L = s$ for the time being. $\mathbf{z}_{t,k}$ is defined by

$$\mathbf{z}_{t,k} = \mathbf{I} - \frac{\mathbf{p}}{2} + \mathbf{h}_{l_1} + \cdots + \mathbf{h}_{l_t} - \mathbf{h}_{s_1} - \cdots - \mathbf{h}_{s_k}, \tag{3.32}$$

where $\mathbf{I} \in \mathcal{J}(m)$ is the angle variable of a path $b = b_1 \otimes \cdots \otimes b_L \in B_{s_1} \otimes \cdots \otimes B_{s_L}$, and $\mathbf{p} = {}^t(p_{j_1}, \ldots, p_{j_g})$. To each edge, either vertical or horizontal, we attach a number $\Theta(\mathbf{z}, \mathbf{z}')$ via the rule:

$$\mathbf{z}' \,\Big|\, \mathbf{z}\ , \quad \begin{array}{c} \mathbf{z} \\ \hline \mathbf{z}' \end{array} \quad \rightarrow \quad \Theta(\mathbf{z}, \mathbf{z}') := \Theta(\mathbf{z}) - \Theta(\mathbf{z}') - \Theta(\mathbf{z} + \mathbf{h}_\infty) + \Theta(\mathbf{z}' + \mathbf{h}_\infty). \tag{3.33}$$

The rule (3.33) can also be described neatly in terms of a two-layer cubic lattice whose sites are assigned with $\mathbf{z}_{t,k}$ or $\mathbf{z}_{t,k} + \mathbf{h}_\infty$. Assign the 2-component integer vectors

$$\begin{aligned} x_{t,k} &= \Big(l_t - \Theta(\mathbf{z}_{t-1,k-1}, \mathbf{z}_{t,k-1}), \Theta(\mathbf{z}_{t-1,k-1}, \mathbf{z}_{t,k-1})\Big), \\ y_{t,k} &= \Big(s_k - \Theta(\mathbf{z}_{t-1,k}, \mathbf{z}_{t-1,k-1}), \Theta(\mathbf{z}_{t-1,k}, \mathbf{z}_{t-1,k-1})\Big) \end{aligned} \tag{3.34}$$

to the edges as follows:

$$\begin{array}{ccccc} & & y_{t,k} & & \\ & \mathbf{z}_{t-1,k-1} & \Big| & \mathbf{z}_{t-1,k} & \\ x_{t,k} & \text{———} & + & \text{———} & x_{t,k+1} \\ & \mathbf{z}_{t,k-1} & \Big| & \mathbf{z}_{t,k} & \\ & & y_{t+1,k} & & \end{array}$$

By the same argument as [23], one can show that the ultradiscrete tau function in [25] for the periodic system coincides essentially with Θ (3.30) here. From this fact and Theorem 3.11 we obtain

Theorem 3.13.

$$x_{t,k} \in B_{l_t},\ y_{t,k} \in B_{s_k}, \quad R(x_{t,k} \otimes y_{t,k}) = y_{t+1,k} \otimes x_{t,k+1}, \tag{3.35}$$

where R denotes the combinatorial R. The path b is reproduced from $\mathbf{I} \in \mathcal{J}(m)$ by

$$b = y_{1,1} \otimes y_{1,2} \otimes \cdots \otimes y_{1,L}.$$

The assertion $x_{t,k} \in B_{l_t}$ means that $\Theta(\mathbf{z}_{t-1,k-1}, \mathbf{z}_{t,k-1})$ is an integer in the range $[0, l_t]$, and similarly for $y_{t,k} \in B_{s_k}$. The periodic boundary condition in the horizontal direction

$$x_{t,0} = x_{t,L}$$

is valid for any t. This can easily be checked by using $\mathbf{z}_{t,0} - \mathbf{z}_{t,L} = \mathbf{h}_{s_1} + \cdots + \mathbf{h}_{s_L} = A^t(1, 1 \ldots, 1)$ and the quasi-periodicity (3.31). Theorem 3.13 tells that $x_{t,k}$ and $y_{t,k}$ obey the local dynamics (combinatorial R) of the generalized periodic box-ball system. As a result, we get

$$T_{l_t} \cdots T_{l_1}(b) = y_{t+1,1} \otimes y_{t+1,2} \otimes \cdots \otimes y_{t+1,L}. \tag{3.36}$$

In this way the solution of the initial value problem under arbitrary time evolutions $\{T_{l_t}\}$ is written down explicitly. Note that we have given an explicit formula not only for $y_{t,k}$ but also $x_{t,k}$ which is often called "carrier" [34].

These aspects of the periodic box-ball system on $\mathcal{P} = B_s^{\otimes L}$ admit a generalization to the inhomogeneous case $\mathcal{P} = B_{s_1} \otimes \cdots \otimes B_{s_L}$. A typical example is a combinatorial version of Yang's system [38, 36], which is endowed with the family of time evolutions $\{T_{s_1}, \ldots, T_{s_L}\}$. The well-known relation $T_{s_1} T_{s_2} \cdots T_{s_L} = \mathrm{Id}$ is understood from $\mathbf{h}_{s_1} + \cdots + \mathbf{h}_{s_L} = A^t(1, 1 \ldots, 1) \in \Gamma$ in our linearization scheme.

To adapt the formalism to the inhomogeneous situation $\mathcal{P} = B_{s_1} \otimes \cdots \otimes B_{s_L}$, we employ (3.23) to specify Φ and replace the vacancy number (3.10) with

$$p_j = \sum_{i=1}^{L} \min(s_i, j) - 2E_j.$$

Under the condition (3.12), we conjecture that Theorem 3.13 remains valid. For an illustration, we consider the path

$$b = 11 \otimes 2 \otimes 1 \otimes 2 \otimes 122 \otimes 1 \otimes 12 \otimes 2 \otimes 1. \tag{3.37}$$

The action-angle variable is depicted as

$$\begin{array}{lll} \boxed{}\boxed{}\boxed{} & -1 & \\ \boxed{}\boxed{} & 1 & \\ \boxed{} & 0 & \end{array}$$

So the vacancy numbers $\mathbf{p}$, the period matrix A and the angle variable $\mathbf{I}$ are given by

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 6 & 4 \\ 2 & 4 & 7 \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

We set $(s_1, \ldots, s_9) = (2,1,1,1,3,1,2,1,1)$ according to (3.37). Let us take $(l_1, l_2, l_3) = (2,1,3)$ and consider the time evolution $b \to T_{l_1}(b) \to T_{l_2}T_{l_1}(b) \to T_{l_3}T_{l_2}T_{l_1}(b)$. Then the edge variables exhibit the following pattern:

$$\begin{array}{ccccccccccccccccccc}
 & 11 & & 2 & & 1 & & 2 & & 122 & & 1 & & 12 & & 2 & & 1 & \\
12 & + & 11 & + & 12 & + & 11 & + & 12 & + & 12 & + & 11 & + & 12 & + & 22 & + & 12 \\
 & 12 & & 1 & & 2 & & 1 & & 122 & & 2 & & 11 & & 1 & & 2 & \\
2 & + & 1 & + & 1 & + & 2 & + & 1 & + & 2 & + & 2 & + & 1 & + & 1 & + & 2 \\
 & 22 & & 1 & & 1 & & 2 & & 112 & & 2 & & 12 & & 1 & & 1 & \\
111 & + & 122 & + & 112 & + & 111 & + & 112 & + & 112 & + & 122 & + & 122 & + & 112 & + & 111 \\
 & 11 & & 2 & & 2 & & 1 & & 112 & & 1 & & 12 & & 2 & & 2 &
\end{array}$$

In terms of the edge variable $\Theta(\mathbf{z}, \mathbf{z}')$ (3.33) representing the number of the letter 2 in tableaux on edges, this looks as

$$\begin{array}{|ccccccccccccccccccc|}
\hline
 & 0 & & 1 & & 0 & & 1 & & 2 & & 0 & & 1 & & 1 & & 0 & \\
1 & + & 0 & + & 1 & + & 0 & + & 1 & + & 1 & + & 0 & + & 1 & + & 2 & + & 1 \\
 & 1 & & 0 & & 1 & & 0 & & 2 & & 1 & & 0 & & 0 & & 1 & \\
1 & + & 0 & + & 0 & + & 1 & + & 0 & + & 1 & + & 1 & + & 0 & + & 0 & + & 1 \\
 & 2 & & 0 & & 0 & & 1 & & 1 & & 1 & & 1 & & 0 & & 0 & \\
0 & + & 2 & + & 1 & + & 0 & + & 1 & + & 1 & + & 2 & + & 2 & + & 1 & + & 0 \\
 & 0 & & 1 & & 1 & & 0 & & 1 & & 0 & & 1 & & 1 & & 1 & \\
\hline
\end{array} \tag{3.38}$$

The table of the values of the ultradiscrete Riemann theta $\Theta(\mathbf{z}_{t,k})$ is as follows:

0	0	1	2	4	8	10	14	17	20
0	0	0	1	2	5	7	10	12	15
0	0	0	0	1	3	5	8	10	12
2	0	0	0	0	1	2	4	6	8

Similarly the values of $\Theta(\mathbf{z}_{t,k} + \mathbf{h}_\infty)$ are given as follows:

0	0	0	1	2	4	6	9	11	14
1	0	0	0	1	2	3	6	8	10
2	0	0	0	0	1	2	4	6	8
4	2	1	0	0	0	1	2	3	4

From these values of $\Theta(\mathbf{z}_{t,k})$ and $\Theta(\mathbf{z}_{t,k} + \mathbf{h}_\infty)$, the table (3.38) is reproduced by the rule (3.33).

4. Summary

We have introduced the local energy distribution of paths in section 2 and reformulated the KKR map ϕ in Theorem 2.2. Combined with the result for ϕ^{-1} [20, 28], it completes the crystal interpretation of the KKR bijection for $U_q(\widehat{\mathfrak{sl}}_2)$.

In section 3 the generalized periodic box-ball system on $B_s^{\otimes L}$ is studied. Under the condition (3.12), the set of paths $\widehat{\mathcal{P}}(m)$ (3.8) characterized by conserved quantities enjoy all the properties (i) and (ii) stated under (3.11). The action-angle variables are introduced in section 3.2. The inverse scattering formalism, i.e., simultaneous linearization of the commuting family of time evolutions, is established in Theorem 3.11. It leads to the formulas for state counting (Theorem 3.12), generic period (3.28), and an algorithm for solving the initial value problem. According to Theorem 3.13, the solution of the initial value problem (3.36) is expressed in terms of the ultradiscrete Riemann theta function (3.30). A similar formula has been obtained also for the carrier variable $x_{t,k}$ simultaneously. These results extend the $s = 1$ case [21, 23] and agree with the conjecture on the most general case [22].

Appendix A. Crystals and Combinatorial R

A.1. **Crystals.** We recapitulate the basic facts in the crystal base theory [10, 11, 14]. Let B_l the the crystal of the l-fold symmetric tensor representation of $U_q(A_1^{(1)})$. As a set it is given by $B_l = \{x = (x_1, x_2) \in (\mathbb{Z}_{\geq 0})^2 \mid x_1 + x_2 = l\}$. The element (x_1, x_2) will also be expressed as the length l row shape semistandard tableau containing the letter i x_i times. For example, $B_1 = \{\boxed{1}, \boxed{2}\}, B_2 = \{\boxed{11}, \boxed{12}, \boxed{22}\}$. (We shall often omit the frames of the tableaux.) The action of Kashiwara operators $\tilde{f}_i, \tilde{e}_i : B \to B \sqcup \{0\}\,(i = 0, 1)$ reads

$(\tilde{f}_i x)_j = x_j - \delta_{j,i} + \delta_{j,i+1}$ and $(\tilde{e}_i x)_j = x_j + \delta_{j,i} - \delta_{j,i+1}$, where all the indices are in $\mathbb{Z}_2$, and if the result does not belong to $(\mathbb{Z}_{\geq 0})^2$, it should be understood as 0. The classical part of the weight of $x = (x_1, x_2) \in B_l$ is $\mathrm{wt}(x) = l\Lambda_1 - x_2\alpha_1 = (x_1 - x_2)\Lambda_1$, where Λ_1 and $\alpha_1 = 2\Lambda_1$ are the fundamental weight and the simple root of A_1.

For any $b \in B$, set

$$\varepsilon_i(b) = \max\{m \geq 0 \mid \tilde{e}_i^m b \neq 0\}, \quad \varphi_i(b) = \max\{m \geq 0 \mid \tilde{f}_i^m b \neq 0\}.$$

By the definition one has $\varepsilon_i(x) = x_{i+1}$ and $\varphi_i(x) = x_i$ for $x = (x_1, x_2) \in B_l$. Thus $\varepsilon_i(x) + \varphi_i(x) = l$ is valid for any $x \in B_l$.

For two crystals B and B', one can define the tensor product $B \otimes B' = \{b \otimes b' \mid b \in B, b' \in B'\}$. The operators $\tilde{e}_i, \tilde{f}_i$ act on $B \otimes B'$ by

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b') \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases} \tag{A.1}$$

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'). \end{cases} \tag{A.2}$$

Here $0 \otimes b'$ and $b \otimes 0$ should be understood as 0. The tensor product $B_{l_1} \otimes \cdots \otimes B_{l_k}$ is obtained by repeating the above rule. The classical part of the weight of $b \in B$ for any $B = B_{l_1} \otimes \cdots \otimes B_{l_k}$ is given by $\mathrm{wt}(b) = (\varphi_1(b) - \varepsilon_1(b))\Lambda_1 = (\varepsilon_0(b) - \varphi_0(b))\Lambda_1$.

Let s_i $(i = 0, 1)$ be the Weyl group operator [10] acting on any crystal B as

$$s_i(b) = \begin{cases} \tilde{f}_i^{\varphi_i(b)-\varepsilon_i(b)}(b) & \varphi_i(b) \geq \varepsilon_i(b), \\ \tilde{e}_i^{\varepsilon_i(b)-\varphi_i(b)}(b) & \varphi_i(b) \leq \varepsilon_i(b) \end{cases} \tag{A.3}$$

for $b \in B$. Let

$$\omega : (x_1, x_2) \mapsto (x_2, x_1) \tag{A.4}$$

be the involutive Dynkin diagram automorphism of B_l. We extend it to any $B = B_{l_1} \otimes \cdots \otimes B_{l_k}$ by $\omega(B) = \omega(B_{l_1}) \otimes \cdots \otimes \omega(B_{l_k})$. Then $\widehat{W}(A_1^{(1)}) = \langle \omega, s_0, s_1 \rangle$ acts on B as the extended affine Weyl group of type $A_1^{(1)}$.

The action of $\tilde{f}_i, \tilde{e}_i$ and s_i is determined in principle by (A.1) and (A.2). Here we explain the *signature rule* to find the action on any $B_{l_1} \otimes \cdots \otimes B_{l_k}$, which is of great practical use. The i-signature of an element $b \in B_l$ is the symbol $\overbrace{- \cdots -}^{\varepsilon_i(b)}\overbrace{+ \cdots +}^{\varphi_i(b)}$. The i-signature of the tensor product $b_1 \otimes \cdots \otimes b_k \in B_{l_1} \otimes \cdots \otimes B_{l_k}$ is the array of the i-signature of each b_j. Here is an example from $B_5 \otimes B_2 \otimes B_1 \otimes B_4$:

$$\begin{array}{rccccccc} & 11112 & \otimes & 12 & \otimes & 2 & \otimes & 1122 \\ 0\text{-signature} & ----+ & & -+ & & + & & --++ \\ 1\text{-signature} & -++++ & & -+ & & - & & --++ \end{array}$$

where 1122 for example represents $\boxed{1122} \in B_4$ and not $\boxed{1} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{2} \in B_1^{\otimes 4}$, etc. In the i-signature, one eliminates the neighboring pair $+-$ (not $-+$) successively to finally reach the pattern $\overbrace{- \cdots -}^{\alpha}\overbrace{+ \cdots +}^{\beta}$ called reduced i-signature. The result is independent of the order of the eliminations when it can be done simultaneously in more than one places. The reduced i-signature tells that $\varepsilon_i(b_i \otimes \cdots \otimes b_k) = \alpha$ and $\varphi_i(b_i \otimes \cdots \otimes b_k) = \beta$. In the above example, we get

$$\begin{array}{rccccccc} & 11112 & \otimes & 12 & \otimes & 2 & \otimes & 1122 \\ 0\text{-signature} & ---- & & & & & & ++ \\ 1\text{-signature} & -+ & & & & & & ++ \end{array}$$

Thus $\varepsilon_0 = 4$, $\varphi_0 = 2$, $\varepsilon_1 = 1$ and $\varphi_1 = 3$. Finally $\tilde{f}_i$ hits the component that is responsible for the leftmost $+$ in the reduced i-signature making it $-$. Similarly, $\tilde{e}_i$ hits the component corresponding to the rightmost $-$ in the reduced i-signature making it $+$. If there is no such

$+$ or $-$ to hit, the result of the action is 0. The Weyl group operator s_i acts so as to change the reduced i-signature $\overbrace{-\cdots-}^{\alpha}\overbrace{+\cdots+}^{\beta}$ into $\overbrace{-\cdots-}^{\beta}\overbrace{+\cdots+}^{\alpha}$. In the above example, we have

$$\begin{aligned}
p &= 11112 \otimes 12 \otimes 2 \otimes 1122 \\
\tilde{f}_0(p) &= 11112 \otimes 12 \otimes 2 \otimes 1112 \\
\tilde{f}_1(p) &= 11122 \otimes 12 \otimes 2 \otimes 1122 \\
\tilde{e}_0(p) &= 11122 \otimes 12 \otimes 2 \otimes 1122 \\
\tilde{e}_1(p) &= 11111 \otimes 12 \otimes 2 \otimes 1122 \\
s_0(p) &= 11222 \otimes 12 \otimes 2 \otimes 1122 \\
s_1(p) &= 11122 \otimes 12 \otimes 2 \otimes 1222.
\end{aligned}$$

For both $i = 0$ and 1, note that $\mathrm{wt}(s_i(p)) = -\mathrm{wt}(p)$ for any p, and $s_i(p) = p$ if $\mathrm{wt}(p) = 0$. In order that $\tilde{e}_1 p = 0$ to hold for any $p \in B_{l_1} \otimes \cdots \otimes B_{l_k}$, it is necessary and sufficient that the reduced 1-signature of p to become $+\cdots+$.

A.2. **Combinatorial R.** The crystal B_l admits the affinization $\mathrm{Aff}(B_l)$. It is the infinite set $\mathrm{Aff}(B_l) = \{b[d] \mid b \in B_l, d \in \mathbb{Z}\}$ endowed with the crystal structure $\tilde{e}_i(b[d]) = (\tilde{e}_i b)[d + \delta_{i,0}]$, $\tilde{f}_i(b[d]) = (\tilde{f}_i b)[d - \delta_{i,0}]$. The element $b[d]$ will also be denoted by $\zeta^d b$ to save the space, where ζ is an indeterminate.

The isomorphism of the affine crystal $\mathrm{Aff}(B_l)\otimes\mathrm{Aff}(B_k) \xrightarrow{\sim} \mathrm{Aff}(B_k)\otimes\mathrm{Aff}(B_l)$ is the unique bijection that commutes with Kashiwara operators (up to a constant shift of H below). It is the $q = 0$ analogue of the quantum R and called the combinatorial R. Explicitly it is given by [15, 37]

$$R(x[d] \otimes y[e]) = \tilde{y}[e - H(x \otimes y)] \otimes \tilde{x}[d + H(x \otimes y)],$$

where $\tilde{x} = (\tilde{x}_i), \tilde{y} = (\tilde{y}_i)$ are given by

$$\begin{aligned}
&\tilde{x}_i = x_i + Q_i(x, y) - Q_{i-1}(x, y), \quad \tilde{y}_i = y_i + Q_{i-1}(x, y) - Q_i(x, y), \\
&Q_i(x, y) = \min(x_{i+1}, y_i), \quad H(x \otimes y) = Q_0(x, y).
\end{aligned}$$

Here $x \otimes y \simeq \tilde{y} \otimes \tilde{x}$ under the isomorphism $B_l \otimes B_k \simeq B_k \otimes B_l$, which is also called (classical) combinatorial R. H is called the local energy function. It is characterized by the recursion relation:

(A.5)
$$H(\tilde{e}_i(x \otimes y)) = \begin{cases} H(x \otimes y) - 1 & \text{if } i = 0,\ \tilde{e}_0(x \otimes y) = \tilde{e}_0 x \otimes y,\ \tilde{e}_0(\tilde{y} \otimes \tilde{x}) = \tilde{e}_0 \tilde{y} \otimes \tilde{x}, \\ H(x \otimes y) + 1 & \text{if } i = 0,\ \tilde{e}_0(x \otimes y) = x \otimes \tilde{e}_0 y,\ \tilde{e}_0(\tilde{y} \otimes \tilde{x}) = \tilde{y} \otimes \tilde{e}_0 \tilde{x}, \\ H(x \otimes y) & \text{otherwise.} \end{cases}$$

together with the connectedness of the crystal $B_l \otimes B_k$. (The original H [11] is $-H$ here.) $Q_0(x, y)$ is a solution of (A.5) normalized so as to attain the minimum at $Q_0(u_l \otimes u_k) = 0$ and ranges over $0 \le Q_0 \le \min(l, k)$ on $B_l \otimes B_k$. Here,

$$u_l = (l, 0) = \boxed{1...1} \in B_l \tag{A.6}$$

denotes the highest element with respect to the sl_2 subcrystal concerning $\tilde{e}_1, \tilde{f}_1$. The invariance $Q_i(x \otimes y) = Q_i(\tilde{y} \otimes \tilde{x})$ holds. When $l = k$, the classical part of the combinatorial R is trivial:

$$R(\zeta^d x \otimes \zeta^e y) = \zeta^{e - H(x\otimes y)} x \otimes \zeta^{d + H(x \otimes y)} y \quad \text{on } B_l \otimes B_l. \tag{A.7}$$

The combinatorial R has the following properties:

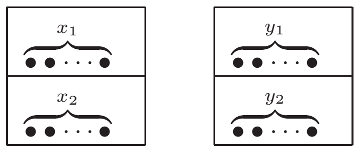
(A.8) $(\omega \otimes \omega)R = R(\omega \otimes \omega)$ on $B_l \otimes B_k$,

(A.9) $R\,\varrho = \varrho\, R$ on $B_l \otimes B_k$,

(A.10)
$$(1 \otimes R)(R \otimes 1)(1 \otimes R) = (R \otimes 1)(1 \otimes R)(R \otimes 1) \quad \text{on } \mathrm{Aff}(B_j) \otimes \mathrm{Aff}(B_l) \otimes \mathrm{Aff}(B_k).$$

Here ω is the involutive automorphism (A.4). $\varrho(b_1 \otimes \cdots \otimes b_k) = b_k \otimes \cdots \otimes b_1$ is the reverse ordering of the tensor product for any k. (A.10) is the Yang-Baxter relation.

To calculate the combinatorial R, it is convenient to use the graphical rule ([14] Rule 3.11). Consider the two elements $x = (x_1, x_2) \in B_k$ and $y = (y_1, y_2) \in B_l$. Draw the following diagram to express the tensor product $x \otimes y$.



The combinatorial R and energy function H for $x \otimes y \in B_k \otimes B_l$ (with $k \geq l$) are found by the following rule.

(i) Pick any dot, say $\bullet_a$, in the right column and connect it with a dot $\bullet'_a$ in the left column by a line. The partner $\bullet'_a$ is chosen from the dots whose positions are higher than that of $\bullet_a$. If there is no such a dot, go to the bottom, and the partner $\bullet'_a$ is chosen from the dots in the lower row. In the former case, we call such a pair "unwinding," and, in the latter case, we call it "winding."

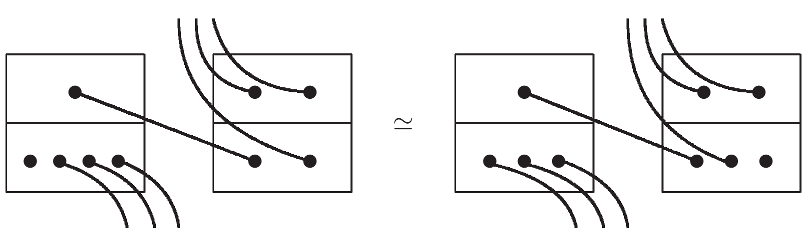
(ii) Repeat procedure (i) for the remaining unconnected dots $(l-1)$ times.

(iii) The isomorphism is obtained by moving all unpaired dots in the left column to the right horizontally. We do not touch the paired dots during this move.

(iv) The energy function H is given by the number of unwinding pairs.

The number of winding (unwinding) pairs is called the winding (unwinding) number. It is known that the result is independent of the order of making pairs ([14], Propositions 3.15 and 3.17). In the above description, we only consider the case $k \geq l$. The other case $k \leq l$ can be done by reversing the above procedure, noticing the fact $R^2 = \mathrm{id}$. For more properties, including that the above definition indeed satisfies the axiom, see [14].

Example A.1. Corresponding to the tensor product $\boxed{12222} \otimes \boxed{1122}$, we draw diagram of the left hand side of the following.



By moving one unpaired dot to the right, we obtain

$$\boxed{12222} \otimes \boxed{1122} \simeq \boxed{1222} \otimes \boxed{11222}.$$

Since we have one unwinding pair, the energy function is $H(\boxed{12222} \otimes \boxed{1122}) = 1$.

Appendix B. Kerov–Kirillov–Reshetikhin bijection

B.1. **Rigged configurations.** The Kerov-Kirillov-Reshetikhin (KKR) bijection gives a one to one correspondence between rigged configurations and highest paths. The latter are elements of $B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_L}$ annihilated by $\tilde{e}_1$. See around (3.1).

Let us define the rigged configurations. Consider a pair (λ, μ), where both λ and μ are positive integer sequences:

$$\begin{aligned} \lambda &= (\lambda_1, \lambda_2, \cdots, \lambda_L), \qquad (L \in \mathbb{Z}_{\geq 0},\ \lambda_i \in \mathbb{Z}_{>0}), \\ \mu &= (\mu_1, \mu_2, \cdots, \mu_N), \qquad (N \in \mathbb{Z}_{\geq 0},\ \mu_i \in \mathbb{Z}_{>0}), \end{aligned}$$

We use usual Young diagrammatic expression for these integer sequences, although λ, μ are not necessarily assumed to be weakly decreasing.

Definition B.1. (1) For a given diagram ν, we introduce coordinates (row, column) of each boxes just like matrix entries. For a box α of ν, $\mathrm{col}(\alpha)$ is the column coordinate of α. Then we define the following subsets:

$$\nu|_{\leq j} := \{\alpha | \alpha \in \nu\,, \mathrm{col}(\alpha) \leq j\}, \quad \nu|_{\geq j} := \{\alpha | \alpha \in \nu\,, \mathrm{col}(\alpha) \geq j\}\,.$$

(2) For the diagrams (λ, μ), we define $Q_j^{(a)}$ $(a = 0, 1)$ by

$$Q_j^{(0)} := \sum_{k=1}^{L} \min(j, \lambda_k), \qquad Q_j^{(1)} := \sum_{k=1}^{N} \min(j, \mu_k), \tag{B.1}$$

i.e., the number of boxes in $\lambda|_{\leq j}$ and $\mu|_{\leq j}$. Then the vacancy number p_j for rows of μ is defined by

$$p_j := Q_j^{(0)} - 2Q_j^{(1)}, \tag{B.2}$$

where j is the width of the corresponding row.

Definition B.2. Consider the following data:

$$\mathrm{RC} := \big(\lambda,\ (\mu, r)\big) = \big((\lambda_i)_{i=1}^{L},\ (\mu_i, r_i)_{i=1}^{N}\big)\,.$$

(1) Calculate the vacancy numbers with respect to the pair (λ, μ). If all vacancy numbers for rows of μ are nonnegative, i.e., $0 \leq p_{\mu_i}, (1 \leq i \leq N)$, then RC is called a *configuration*.

(2) If the integer r_i satisfies the condition

$$0 \leq r_i \leq p_{\mu_i}, \tag{B.3}$$

then r_i is called a *rigging* associated with the row μ_i. For the rows of equal widths, i.e., $\mu_i = \mu_{i+1}$, we assume that $r_i \leq r_{i+1}$.

(3) If RC is a configuration and if all integers r_i are riggings associated with row μ_i, then RC is called $\mathfrak{sl}_2$ rigged configuration.

In the rigged configuration, λ is sometimes called a quantum space which determines the shape of the corresponding path, as we will see in the next subsection. In the diagrammatic expression of rigged configurations, the riggings are attached to the right of the corresponding row.

Definition B.3. For a given rigged configuration, consider a row μ_i and the corresponding rigging r_i. If they satisfy the condition $r_i = p_{\mu_i}$, then the row μ_i is called *singular*.

B.2. **Definition of the KKR bijection.** Here we explain the original combinatorial procedure to obtain a rigged configuration RC

$$\phi : b \longrightarrow \mathrm{RC} = \big(\lambda,\ (\mu, r)\big)$$

from a given path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B_{\lambda_1} \otimes B_{\lambda_2} \otimes \cdots \otimes B_{\lambda_L}$. The appearance of λ_i in the right hand side is clear from the following definition.

Definition B.4. Under the above setting, the image of the KKR map ϕ is defined by the following procedure.

(i) We start from the empty rigged configuration $\mathrm{RC}_0 := \big(\emptyset, (\emptyset, \emptyset)\big)$ and construct $\mathrm{RC}_1, \cdots, \mathrm{RC}_{|\lambda|}$ successively as follows (note that $|\lambda| = \sum \lambda_i$).

(ii) Set $b_{1,0} := b_1 \in B_{\lambda_1}$ for the path $b = b_1 \otimes \cdots$. From $b_{1,0}$, we recursively construct $b_{1,1}, b_{1,2}, \cdots, b_{1,\lambda_1}$ and $\mathrm{RC}_1, \mathrm{RC}_2, \cdots, \mathrm{RC}_{\lambda_1}$. $b_{1,i+1}$ and RC_{i+1} are constructed from $b_{1,i} := (x_1, x_2) \in B_{\lambda_1 - i}$ and RC_i as follows:

  (a) First, assume that $x_2 = 0$. Then we set $b_{1,i+1} = (x_1 - 1, 0)$. If $i = 0$, we create a new row to the quantum space as follows:
$$\mathrm{RC}_1 = \big(1, (\emptyset, \emptyset)\big).$$
  If $i > 0$, then we add one box to the row of the quantum space which is lengthened when we construct RC_i.

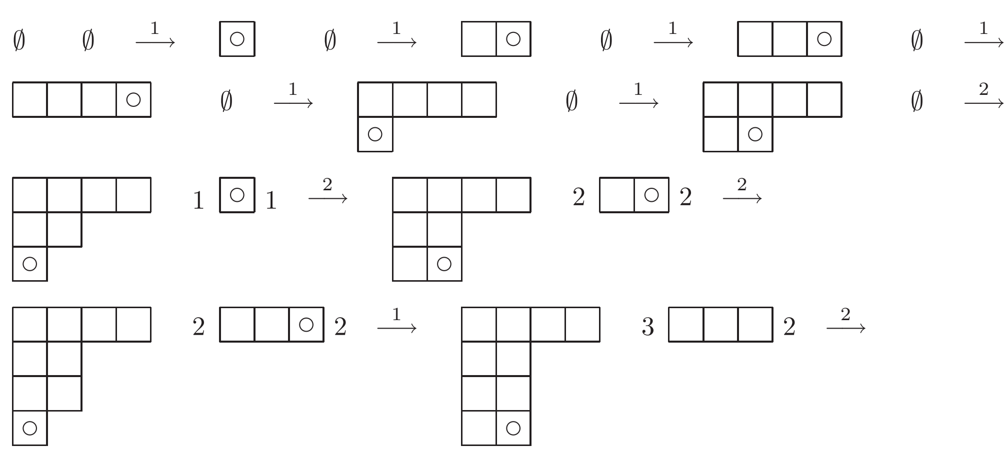
  (b) On the contrary, assume that $x_2 > 0$. Then we set $b_{1,i+1} = (x_1, x_2 - 1)$. We add a box to the quantum space by the same procedure as in the case $x_2 = 0$. Operation on (μ, r) part of RC_i is as follows. Calculate the vacancy numbers of RC_i and determine all the singular rows. If there is no singular row in $\mu|_{\geq i}$, then create a new row in μ. On the other hand, assume that there is at least one singular row in $\mu|_{\geq i}$. Then, among these singular rows, we choose one of the longest singular rows arbitrary, and let us tentatively call it μ_s. We add one box to the row μ_s and do not change the other parts, which gives μ of new RC_{i+1}. As for the riggings, calculate the vacancy numbers of RC_{i+1}. Then we choose r_s, i.e., the rigging associated to the lengthened row μ_s, so as to make the row μ_s singular in RC_{i+1}. Other riggings are chosen to be the same as the corresponding ones in RC_i.

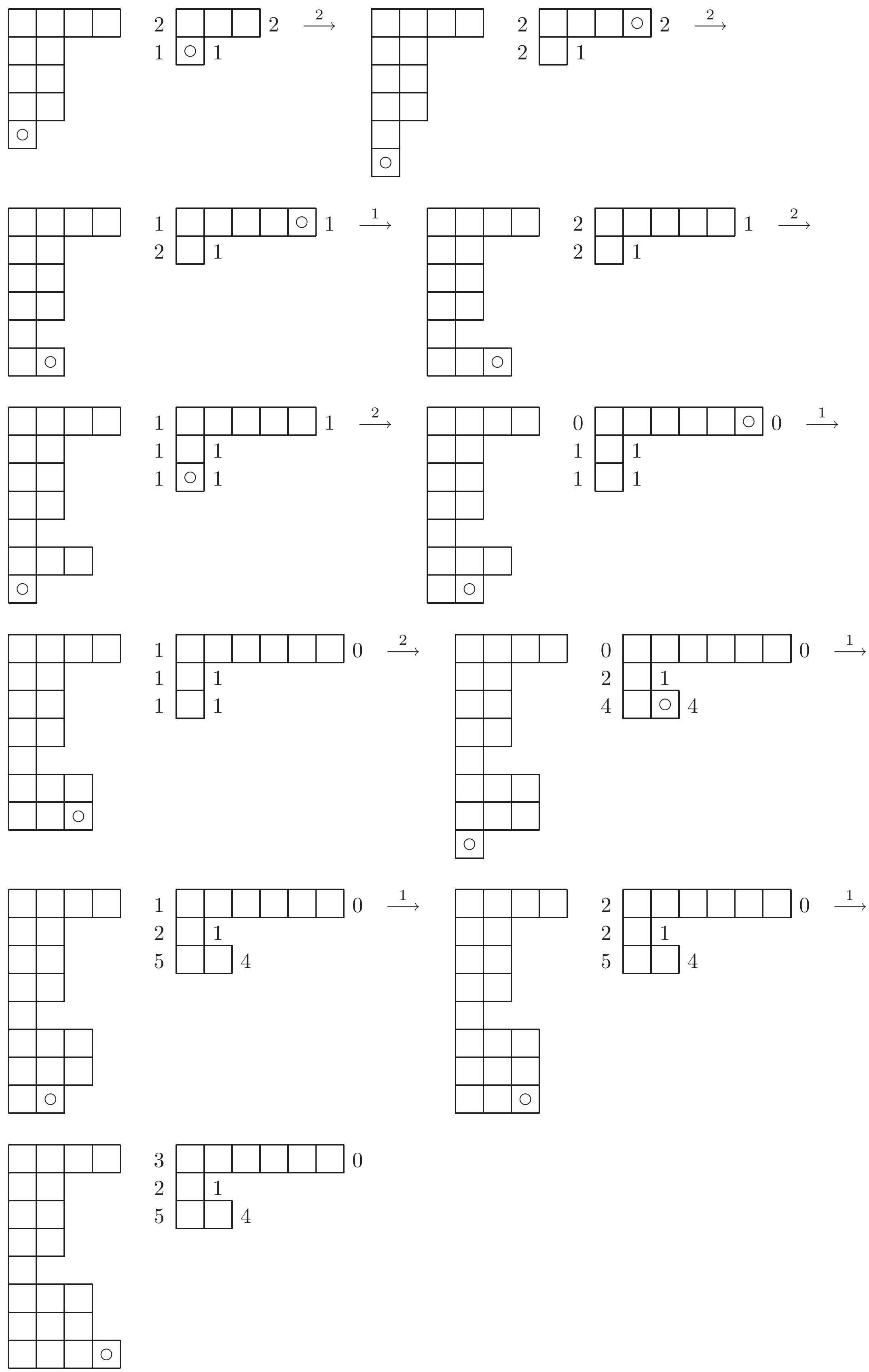
  (c) Repeat the above Step (b) for all letters 2 contained in b_1, then we do Step (a) for the rest of letters 1 in b_1.

(iii) Do the same procedure of Step (ii) for $b_2, \cdots, b_L$. Each time when we start with a new element b_i, we create a new row in the quantum space, and apply Step (ii). The result gives the image RC=$\mathrm{RC}_{|\lambda|}$ of the map ϕ.

It is known that all RC_i in the above procedure are rigged configurations.

Example B.5. Consider the following path:
$$b = \boxed{1111} \otimes \boxed{11} \otimes \boxed{22} \otimes \boxed{12} \otimes \boxed{2} \otimes \boxed{122} \otimes \boxed{122} \otimes \boxed{1112}$$
From the above path, we obtain the sequence of letters $1111 \cdot 11 \cdot 22 \cdot 21 \cdot 2 \cdot 221 \cdot 221 \cdot 2111$. Then the calculation of $\phi(b)$ proceeds as follows.




2 2 2 2 2 2
1 1 2 1
1 1 1 2 2 1 2
2 1 2 1
1 1 2 0 0 1
1 1 1 1
1 1 1 1
1 0 2 0 0 1
1 1 2 1
1 1 4 4
1 0 1 2 0 1
2 1 2 1
5 4 5 4
3 0
2 1
5 4

In the above diagrams, newly added boxes are indicated by circles "○", and vacancy numbers are listed on the left of the corresponding rows in order to facilitate the calculations. This example, fully worked out here, will be revisited in Example 2.3 by using Theorem 2.2 for comparison.

B.3. **Basic properties of the KKR bijection.** It is known that the inverse map ϕ^{-1} can be described by a similar combinatorial procedure. We will use both ϕ and ϕ^{-1} in later arguments.

Theorem B.6. *The inverse map*

$$\phi^{-1} : \mathrm{RC} = \big(\lambda,\, (\mu, r)\big) \longrightarrow b,$$

is obtained by the following procedure ($\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_L)$).

(i) *We construct $RC_{|\lambda|}$, $RC_{|\lambda|-1}$, $\cdots$, RC_1, $RC_0 = \big(\emptyset, (\emptyset, \emptyset)\big)$ and b_L, b_{L-1}, $\cdots$, b_1 ($b_i \in B_{\lambda_i}$) as follows.*

(ii) *We start from λ_L of λ, and set $\mathrm{RC}_{|\lambda|} := \mathrm{RC}$ and $b_{L,\lambda_L} = (0, 0)$. In order to obtain the quantum space of $RC_{|\lambda|-1}$, $RC_{|\lambda|-2}$, $\cdots$, we remove boxes of row λ_L of the quantum space one by one. RC_{i-1} and $b_{L,i-1}$ are constructed from RC_i and $b_{L,i}$. We call the rightmost box of the row of length $\lambda_L - i$ in the quantum space as α. ($\mathrm{col}(\alpha) = \lambda_L - i$.) Let us tentatively denote the "μ part" of RC_i by ν.*

  (a) *Assume that there is no singular row in $\nu|_{\geq \mathrm{col}(\alpha)}$. Then RC_{i-1} is obtained by removing the box α from the quantum space. In this case, $b_{L,i-1}$ is obtained by adding letter 1 to $b_{L,i}$.*

  (b) *Assume on the contrary that there are singular rows in $\nu|_{\geq \mathrm{col}(\alpha)}$. Among these singular rows, we choose one of the shortest rows arbitrary, and denote the rightmost box of the chosen row by β. Then RC_{i-1} is obtained by removing the two boxes α and β from RC_i. New riggings are specified as follows. For the row from which the box β is removed, take new rigging so that it becomes singular in RC_{i-1}. On the contrary, for the other rows riggings are kept unchanged from the corresponding ones in RC_i. Finally, $b_{L,i-1}$ is obtained by adding letter 2 to $b_{L,i}$.*

  (c) *By doing Steps (a) and (b) for the rest of boxes of λ_L in the quantum space, we obtain $b_L \in B_{\lambda_L}$. Here, the orderings of letters 1 and 2 within b_L is chosen so that b_L becomes semi-standard Young tableau.*

(iii) *By doing Step (ii) for the rest of rows λ_{L-1}, $\cdots$, λ_1, we obtain b_{L-1}, $\cdots$, b_1 respectively. Finally, we obtain the path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L$ as the image of the map ϕ^{-1}.*

Example B.7. For an example of the calculation of ϕ^{-1}, one can use Example B.5, that is, reverse all arrows "$\longrightarrow$" to "$\longleftarrow$".

The above map ϕ^{-1} depends on ordering of the quantum space $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_L)$. The dependence is described by

Theorem B.8 ([18] Lemma 8.5)**.** *Let α, β be any successive two rows in the quantum space of a rigged configuration. Suppose the removal of α first and β next by ϕ^{-1} lead to the tableaux a_1 and b_1, respectively. Suppose similarly that the removal of β first and α next lead to b_2 and a_2, respectively. If the order of the other removal is the same,*

$$b_1 \otimes a_1 \,\simeq\, a_2 \otimes b_2$$

is valid under the isomorphism of the combinatorial R.

The KKR bijection $\phi^{\pm 1}$, originally designed only for highest paths, is known to admit an extension which covers *all* the paths. In fact one can apply the same combinatorial procedure as ϕ to obtain $\phi(b)$ for any $b \in B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_L}$. The resulting object is a *unrestricted rigged configuration*, where the condition (B.3) is relaxed to $-\mu_i \leq r_i \leq p_{\mu_i}$

[31, 5]. Obviously rigged configurations are special case of unrestricted ones. For unrestricted rigged configurations, combinatorial procedure in Theorem B.6 also works to define the inverse map ϕ^{-1}. Let us write $\phi(b) = (\lambda, (\mu, r))$. Then, $|\lambda|$ represents the total number of letters 1 and 2 contained in the path b, whereas $|\mu|$ is the number of letter 2 in b. Note in particular that $|\lambda| \geq |\mu|$ holds for unrestricted rigged configurations.

Given a non-highest path b, one can always make

$$b' := \boxed{1}^{\otimes \Lambda} \otimes b$$

highest by taking $\Lambda \geq \lambda_1 + \cdots + \lambda_L$. Under these notations, we have

Lemma B.9. *Let the unrestricted rigged configuration corresponding to b be*

$$\big((\lambda_i)_{i=1}^{L}, (\mu_j, r_j)_{j=1}^{N}\big). \tag{B.4}$$

Then the rigged configuration corresponding to the highest path b' is given by

$$\big((\lambda_i)_{i=1}^{L} \cup (1^{\Lambda}), (\mu_j, r_j + \Lambda)_{j=1}^{N}\big). \tag{B.5}$$

Proof. Let the vacancy number of a row μ_j of the pair (λ, μ) of (B.4) be p_{μ_j}. Then the vacancy number of the row μ_j of (B.5) is equal to $p_{\mu_j} + \Lambda$. Now we apply ϕ^{-1} on (B.5). From the quantum space $\lambda \cup (1^{\Lambda})$, we remove λ first, and next remove (1^{Λ}). Recall that the combinatorial procedure in Theorem B.6 only refers to co-rigging (:=vacancy number − rigging), rather than the riggings. Therefore, when we remove λ from the quantum space of (B.5), we get b as the corresponding part of the image. Then, the remaining rigged configuration has the quantum space (1^{Λ}) without μ part. Since the map ϕ^{-1} becomes trivial on it, we obtain b' as the image of (B.5).

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